The Cuntz Semigroup as a Classification Functor for C^*-Algebras

by

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Abstract

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The classification of C^*-algebras with K-theoretic invariants has recently been found to have approximately homogeneous (AH) counterexamples. These counterexamples were demonstrated to not be isomorphic through the use of the Cuntz semigroup. Moreover, a property in the Cuntz semigroup (called "almost unperforation") has been found to correspond with the "slow dimension growth" property which constitutes something of a best guess at what might identify which C^*-algebras can be classified K-theoretically.

The theory of the Cuntz semigroup still being rather limited, this thesis sets out to put the Cuntz semigroup in a framework where it respects inductive limits. More specfically, a natural category for the Cuntz semigroup is defined, the operation of taking the Cuntz semigroup of a C^*-algebras is shown to be a functor from the category of C^*-algebras to this new category, and this functor is shown to respect inductive limits. A few properties of this new category are also proven.

Additionally, a subsemigroup of the Cuntz semigroup is taken for commutative algebras, and those commutative algebras for which this subsemigroup is the entire Cuntz semigroup are identified.

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Chapter 1

Introduction

Throughout the history of mathematics, there have been efforts to find similarities between various mathematical objects, and work on identifying properties which can simplify the task of testing for these similarities. In recent times (relative to the history of mathematics), there has been a particular interest in the similarities that can be described as isomorphism, and in classification theories which can test for this similarity. Some particularly strightforward examples of this work are the classification of a vector space by its dimension, and of a free abelian group by its rank. The broad concern of this thesis, is the classification theory of C^*-algebras.

The classification theory of C^*-algebras has its origins in the theory of von Neumann algebras (all of which are, in fact, C^*-algebras). In particular, Murray and von Neumann proposed a notion of equivalence for projections (self-adjoint idempotents), where two projections p and q were said to be equivalent (which will be denoted $p \sim_V q$) iff there exists an element v in the algebra, satisfying $p = v^*v$ and $q = vv^*$.

A C^*-algebra is a self-adjoint *-subalgebra of bounded linear operators on a Hilbert space, closed in the operator norm. The notion of Murray-von Neumann equivalence of projections continues to make sense in the more general setting of a C^*-algebra, and provides the basis of a classification theory, using K-theory, that has only recently been shown by Toms (in [17]) to fail to distinguish between non-isomorphic C^*-algebras having certain properties.

Nevertheless, the K-theoretic invariants that have largely motivated the development of the Cuntz invariant, also being simpler objects, still warrant description here. First, this description will require use of the notation

$$M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A)$$

where $M_n(A)$ is identified with the top left $n \times n$ block of $M_m(A)$ for $m > n$.

This allows the Murray-von Neumann semigroup $V(A)$ to be described as the set of Murray-von Neumann equivalence classes of projections in $M_\infty(A)$, with direct summation as the addition operation for the semigroup. $K_0(A)$ is then just the Grothendieck enveloping group for $V(A)$.

The Cuntz semigroup, $W(A)$, was introduced in [3] as a sort of expansion of the Murray-von Neumann semigroup, producing an expanded (at least in the case where A is unital and stably finite) analogue of $K_0(A)$.

Interest in the Cuntz semigroup has been recently rekindled by work by Rørdam in [16], comparing order properties in $W(A)$ with stability under tensor multiplication with the Jiang-Su algebra $\mathcal{Z}$, and with the property of slow dimension growth in approximately homogeneous (AH) C^*-algebras. Further interest has been generated by Toms' use of the Cuntz semigroup to establish the lack of isomorphism between the C^*-algebras he found to have isomorphic K-theoretic invariants.

The Cuntz semigroup is constructed by considering positive elements in $M_\infty(A)$, rather than just projections, and taking equivalence classes as follows:

First, two positive elements a and b will be said to satisfy the preorder relation $a \precsim b$ when there are sequences $\{x_n\}$ and $\{y_n\}$ in $M_\infty(A)$ satisfying

$$a = \lim_{n\to\infty} x_n b y_n$$

(in the norm topology). This relation was later proven by Rørdam, in Proposition 2.4 of [15] to have an equivalent definition that $a \precsim b$ when there is a single sequence $\{x_n\}$ in $M_\infty(A)$ satisfying

$$a = \lim_{n\to\infty} x_n b x_n^*$$

(also in the norm topology). Two positive elements a and b are then said to be equivalent (denoted $a \sim_W b$) when $a \precsim b$, and $b \precsim a$.

It's worth noting here, that the use of a preorder relation, to define Cuntz equivalence of positive elements, induces an order relation in $W(A)$ beyond the one arising from addition. (In fact, Toms and Perera have shown in [14] — in the case that A has stable rank one — that an equivalence class which is majorized by the same classes under either order relation, is the equivalence class of a projection!) It's also worth noting that if you have two projections, p and q, which are Murray-von Neumann equivalent (in particular, when $p = v^*v$ and $q = vv^*$), then they are also Cuntz equivalent (as $v^*qv = v^*vv^*v = pp = p$ and $vpv^* = vv^*vv^* = qq = q$), though the converse does not necessarily hold (for example in the Cuntz algebras $\mathcal{O}_n, n \geq 2$, all projections are Cuntz equivalent).

A technical detail which arises when comparing the Murray-von Neumann and Cuntz semigroups is that of selecting the appropriate *-algebra to select positive elements or projections from. Because any projection in the C^*-algebra of compact operators $\mathcal{K}$ has finite rank, it doesn't particularly matter whether the Murray-von Neumann semigroup is constructed from equivalence classes of projections in $M_\infty(A)$, or from equivalence classes of projections in the tensor product $\mathcal{K} \otimes A$ (the stabilization of A).

This is not the case with the Cuntz semigroup. In fact in $\mathcal{K}$ itself, infinite rank elements, such as $\mathrm{diag}(1, 1/2, 1/3, \ldots)$ form a Cuntz equivalence class of their own. Further, as the direct sum of any infinite rank positive operator with any other positive operator will have infinite rank, we get (noting that $\mathcal{K} \otimes \mathbb{C} \cong \mathcal{K}$) that $W(\mathbb{C}) \cong \mathbb{N}$ if we take positive elements in $M_\infty(\mathbb{C})$, but if we take our positive elements instead from $\mathcal{K} \otimes \mathbb{C}$, then we get $W(\mathbb{C}) \cong \mathbb{N} \cup \infty$.

Given that, in $\mathcal{K}$, the class of infinite rank positive operators absorbs all the other classes under addition, the selection of the classes of positive elements in $\mathcal{K} \otimes A$, as semigroup elements, does not lend itself well to use in an abelian group. Indeed, with such a selection, the Grothendieck enveloping group of $W(\mathbb{C})$ (in fact, $W(A)$, for any C^*-algebra A) would be trivial.

In light of this, the historical decision to define $W(A)$ on $M_\infty(A)$ would seem to be the obvious choice. That said, much of the work contained here actually makes more sense in the context of defining $W(A)$ on $\mathcal{K} \otimes A$ (though when this occurs, the notation $W(\mathcal{K} \otimes A)$ will be used). Alternately, it makes sense to simply view this work as ignoring the case of non-stable C^*-algebras, and working entirely in the setting of stable C^*-algebras.

In particular, the main result of this thesis, at the end of Chapter 5, uses an alternate formulation for the Cuntz semigroup (also developed in Chapter 5) which is isomorphic to $W(\mathcal{K} \otimes A)$ rather than to $W(A)$. The proof of this isomorphism makes up Chapter 6.

Chapter 2 provides background material from other sources that is used to justify results in the later chapters.

Chapter 3 begins an examination of the Cuntz semigroup of commutative algebras, the completion of which uses the alternate formulation described in Chapter 5, and is provided in Chapter 7.

Chapter 4 defines the appropriate category to consider the Cuntz semigroup as an object in, and establishes the existence of inductive limits in this category.

Finally, Chapter 8 describes directions for possible further research, including in particular, the use of preservation of inductive limits by the Cuntz semigroup functor in this new framework, as a mechanism for computing the Cuntz semigroup of inductive limit C^*-algebras which have previously resisted such computations.

Note also that the material in Chapters 4, 5, and 6 is joint work with George Elliott and Cristian Ivanescu.

Chapter 2

Background and essential results

In [8], Ho proved a result on approximate unitary equivalence of certain positive elements that is crucial to the description of (a particularly interesting subsemigroup of) the Cuntz semigroup of commutative C^*-algebras (this result has been revised in [5]). Another important result, due to Kirchberg and Rørdam, provides a description of a positive element, approximating another positive element, in a form that is very useful for describing the Cuntz semigroup in terms of Hilbert C^*-modules (rather than positive elements), and also in some subsequent results.

Both Ho's result, and Kirchberg and Rørdam's result, are presented here, as well as some material on Hilbert C^*-modules.

2.1 An approximate unitary equivalence for $C(X)$

Before proceeding with this result, it may be helpful to define the notion of approximate unitary equivalence between two elements of a C^*-algebra.

Definition Two elements a, b of a C^*-algebra A are said to be approximately unitarily equivalent if there exists a sequence $\{u_n\}$ of unitaries in A (or in its unitization, if A is not unital) satisfying $||u_n^* a u_n - b|| \to 0$.

Given a continuous diagonal matrix valued function over a topological space X (that is, a diagonalized element of $M_n(C(X))$), it is occasionally useful to rearrange the entries on the diagonal on some open subset of X. Under certain conditions, this can be done:

Lemma 2.1 (Ho). *Given $f = diag(\lambda_1(x), \ldots, \lambda_n(x)) \in M_n(C(X))$, ($\lambda_i \in C(X)$ for each i), Y an open subset of X, $1 \leq j < k \leq n$, and $\lambda_j(y) = \lambda_k(y) foreveryy$ in the boundary of Y, then f is approximately unitarily equivalent to $g = diag(\lambda_1(x), \ldots, \lambda_k(x), \ldots, \lambda_j(x), \ldots, \lambda_n(x))$.*

Proof. Let $\{U_n\}$ be an increasing sequence of closed sets contained in Y, whose union is Y itself (in particular, take $U_n = \{y \in Y; |\lambda_j(y) - \lambda_k(y)| \geq 1/n\}$), and let $\{V_n\}$ be a similar sequence of open sets covering the complement of the closure of Y. Now take $v(0)$ to be the identity on M_n, $v(1)$ to be the matrix that exchanges the jth and kth elements, and $v : [0, 1] \to M_n(\mathbb{C})$ a path of unitaries joining $v(0)$ and $v(1)$. Now let h_n be a continuous $[0, 1]$-valued function on X, equal to 0 on V_n and 1 on U_n. Finally, take u_n to be the isomorphic image in $M_n(C(X))$ of $v \circ h_n$ (in $C(X, M_n(\mathbb{C}))$).

Now the unitary path v is can be chosen so that any unitary $v(i)$ along that path will satisfy $||v(i)^* m v(i) - v(e)^* m v(e)|| \leq |\lambda(j) - \lambda(k)|$ for $e \in \{0, 1\}$. Because u_n differs from the desired limit operator only on regions where $|\lambda_j(x) - \lambda_k(x)| < 1/n$, we get that $||u_n^* f u_n - g|| < 1/n$, so g is approximately unitarily equivalent to f. □

2.2 A useful approximation

This result by Kirchberg and Rørdam is taken from [10], and requires some background of its own (which is taken from the same, except where noted).

First, the notation needs to be introduced for a positive $a \in A$ that $(a - \epsilon)_+$ is the positive part of the self-adjoint element $(a - \epsilon)$ (note that when A is not unital, ϵ needs to be taken from the unitization of A). This can be taken two ways, both of which give that $(a-\epsilon)_+ \in A$ (and more importantly, give the same value); the first is simply to apply functional calculus to $(a-\epsilon)$, using the function $h_\epsilon : \mathbb{R} \to \mathbb{R}^+$ defined by $h(t) = \max(t-\epsilon, 0)$. The other is simply to take $(a-\epsilon)_+ = \frac{1}{2}[(a-\epsilon) + |a-\epsilon|]$ (noting that this still uses functional calculus to take the square root of $(a-\epsilon)^*(a-\epsilon)$. It should also be noted that:

$$(a-\epsilon_1-\epsilon_2)_+ = \big((a-\epsilon_1)_+ - \epsilon_2\big)_+, \quad \text{and} \quad ||(a-\epsilon)_+ - a|| \leq \epsilon$$

hold for all $a \in A^+$ and all $\epsilon, \epsilon_1, \epsilon_2 \geq 0$.

Additionally, we will need the polar decomposition; that every element x in a C^*-algebra A has a polar decomposition $x = u|x|$, where u is a partial isometry in the enveloping von Neumann algebra A^{**} (where A^* is the commutant of A, i.e. the C^*-algebra of bounded operators on Hilbert space which commute with every element of A, when A is considered as a subalgebra of the algebra of bounded operators on Hilbert space). Additionally, for all elements y in the hereditary sub-C^*-algebra $\overline{x^*Ax}$ (of A^{**}), the elements uy, yu^*, and uyu^* belong to A. Also, the mapping $y \mapsto uyu^*$ defines an isomorphism from $\overline{x^*Ax}$ onto $\overline{xAx^*}$.

To prove the approximation result, we also need the following two results from [12]

Lemma 2.2 (Kirchberg, Rørdam). *Let x, y and a be elements of a C^*-algebra A such that $a \geq 0$ and $x^*x \leq a^\alpha$, $yy^* \leq a^\beta$ with $\alpha + \beta \geq 1$. Then the sequence with elements $u_n = x(\frac{1}{n} + a)^{-1/2}y$ is norm convergent to an element u in A with $||u|| \leq ||a^{(\alpha+\beta-1)/2}||$.*

Proof. Put $d_{nm} = (\frac{1}{n} + a)^{-1/2} - (\frac{1}{m} + a)^{-1/2}$. Then

$$\begin{aligned} ||u_n - u_m||^2 &= ||xd_{nm}y||^2 = ||y^*d_{nm}x^*xd_{nm}y|| \\ &\leq ||y^*d_{nm}a^\alpha d_{nm}y|| = ||a^{\alpha/2}d_{nm}y||^2 \\ &= ||a^{\alpha/2}d_{nm}yy^*d_{nm}a^{\alpha/2}|| \\ &\leq ||a^{\alpha/2}d_{nm}a^\beta d_{nm}a^{\alpha/2}|| = ||d_{nm}a^{(\alpha+\beta)/2}||^2. \end{aligned}$$

From spectral theory, we see that the sequence $\{(\frac{1}{n}+a)^{-1/2}a^{(\alpha+\beta)/2}\}$ is increasing and thus by Dini's theorem uniformly convergent to $a^{(\alpha+\beta-1)/2}$. Consequently $||d_{nm}a^{(\alpha+\beta)/2}|| \to 0$ so that $\{u_n\}$ is norm convergent to an element u in A. We have

$$||u_n|| = ||x(\frac{1}{n}+a)^{-1/2}y|| \leq ||a^{\alpha/2}(\frac{1}{n}+a)^{-1/2}a^{\beta/2}|| \leq ||a^{(\alpha+\beta-1)/2}||,$$

reasoning as above; which shows that $||u|| \leq ||a^{(\alpha+\beta-1)/2}||$. □

Proposition 2.3 (Kirchberg, Rørdam). *Let x and a be elements in a C^*-algebra A such that $a \geq 0$ and $x^*x \leq a$. If $0 < \alpha < \frac{1}{2}$ there is an element u in A with $||u|| \leq ||a^{\frac{1}{2}-\alpha}||$ such that $x = ua^\alpha$.*

Proof. Define $u_n = x(\frac{1}{n}+a)^{-\frac{1}{2}}a^{\frac{1}{2}-\alpha}$. From Lemma 2.2 we see that $\{u_n\}$ is convergent to an element u in A with

$$||u|| \leq ||a^{\frac{1}{2}(1+1-2\alpha-1)}|| = ||a^{\frac{1}{2}-\alpha}||.$$

Furthermore

$$\begin{aligned} ||x-u_na^\alpha||^2 &= ||x[1-(\frac{1}{n}+a)^{-1/2}a^{1/2}]||^2 \\ &\leq ||a^{1/2}[1-(\frac{1}{n}+a)^{-1/2}a^{1/2}]||^2 \to 0 \end{aligned}$$

as $n \to \infty$ by spectral theory (Dini's theorem). It follows that $x = ua^\alpha$. □

Now we can proceed to Kirchberg and Rørdam's most useful (for this thesis) result:

Lemma 2.4 (Kirchberg, Rørdam). *Let A be a C^*-algebra, let a, b be positive elements in A, and let $\epsilon > ||a-b||$ be given. Then there is a contraction d in A such that $dbd^* = (a-\epsilon)_+$.*

Proof. For each $r > 1$ define $g_r : \mathbb{R}^+ \to \mathbb{R}^+$ by $g_r(t) = \min(t, t^r)$. Observe that $g_r(b) \to b$ as $r \to 1$. Choose $r > 1$ such that $(\epsilon_1 =)\ ||a - g_r(b)|| < \epsilon$ and set $b_0 = g_r(b)$. Then $b_0 \leq b$,

$b_0 \leq b^r$, and $a - \epsilon_1 \leq b_0$. Find a positive contraction e in $C^*(a)$ with $e(a-\epsilon_1)e = (a-\epsilon)_+$. Then $(a-\epsilon)_+ \leq eb_0e$. Put $x = b_0^{1/2}e$ and let $x = v|x|$ be the polar decomposition for x, where v is a partial isometry in A^{**}. As $(a-\epsilon)_+ \leq eb_0e = x^*x$ (and is therefore in the hereditaty sub-C^*-algebra $\overline{x^*Ax}$), the element $y = v(a-\epsilon)_+^{1/2}$ belongs to A. $y^*y = (a-\epsilon)_+$, and

$$yy^* = v(a-\epsilon)_+v^* \leq vx^*xv^* = xx^* = b_0^{1/2}e^2b_0^{1/2} \leq b_0.$$

Now, following the proof of Proposition 2.3, put $d_n = y^*(\frac{1}{n} + b^r)^{-1/2}b^{(r-1)/2}$. Because $yy^* \leq b_0 \leq b^r$, Lemma 2.2 applies (with $\alpha = 1$ and $\beta = (r-1)/r$) and shows that $\{d_n\}_{n=1}^{\infty}$ is a Cauchy sequence in A. Let d be the limit of this Cauchy sequence. As in the proof of Proposition 2.3, we have $db^{1/2} = y^*$, so that $dbd^* = y^*y = (a-\epsilon)_+$. Since $yy^* \leq b_0 \leq b$ we get

$$d_n^*d_n \leq b^{(r-1)/2}(\frac{1}{n} + b^r)^{-1/2}b(\frac{1}{n} + b^r)^{-1/2}b^{(r-1)/2} \leq 1.$$

Hence $||d_n|| \leq 1$ for each n which entails that d is a contraction. □

2.3 Hilbert C^*-modules

Given that the last result is to be used in formulating the Cuntz semigroup in terms of Hilbert C^*-modules, it will be helpful in making such a formulation, to have some background on Hilbert C^*-modules. Naturally, we begin by defining what a Hilbert C^*-module is.

Definition Given a C^*-algebra A, a module E over A is a Hilbert C^*-module over A (also called a Hilbert A-module) iff it is equipped with an A-valued inner product satisfying the conditions:

1. $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for every $x, y, z \in E; \alpha, \beta \in \mathbb{C}$

2. $\langle x, ya\rangle = \langle x, y\rangle a$ for every $x, y \in E; a \in A$

3. $\langle y, x\rangle = \langle x, y\rangle^*$ for every $x, y \in E$

4. $\langle x, x\rangle \geq 0$ for every $x \in E$

5. $\langle x, x\rangle = 0$ iff $x = 0$

and is also complete under the norm $||x|| = ||\langle x, x\rangle^{1/2}||$.

It then stands to reason that we can consider A as a module over itself, assign it an inner product $\langle a, b\rangle = a^*b$ and have that it is in fact a Hilbert A-module. Likewise, the free module $A \oplus A$ can be made a Hilbert A-module with the inner product $\langle a_1 \oplus a_2, b_1 \oplus b_2\rangle = a_1^*b_1 + a_2^*b_2$, and make similar constructions for any finite direct sums of A.

Definition Carrying this construction further, we can define $\mathcal{H}_A$ to be the module consisting of square summable series in $\bigoplus^\infty A$, that is those $\{a_i\}$ for which the series $\sum_i a_i^* a_i$ is convergent in A.

This construction then admits the following theorem, due to Kasparov in [9], which will be reproduced here without proof.

Theorem 2.5 (Kasparov). *If A is an algebra with a continuous group action and E is a countably generated Hilbert A-module, then $E \oplus \mathcal{H}_A \cong \mathcal{H}_A$.*

The remaining material on Hilbert C^*-modules to be used in this thesis follows, and is from [11] by Lance.

Our first task here is to define a adjointable endomorphism on a Hilbert A-module E. We say that $t : E \to E$ is adjointable iff there is a map t^* satisfying

$$\langle tx, y\rangle = \langle x, t^*y\rangle \qquad x, y \in E$$

Noting that t is necessarily A-linear, and that it necessarily has an (adjointable) adjoint. For each x in the unit ball E_1 of E, we can define a function $f_x : E \to A$ by $f_x(y) = \langle tx, y\rangle$. Then $||f_x(y)|| \leq ||t^*y||$ for all $x \in E_1$. It follows from the Banach-Steinhaus theorem that the set $\{||f_x|| : x \in E_1\}$ is bounded, and this shows that t is bounded.

Referring to the $*$-algebra of adjointable endomorphisms on E as $\mathcal{L}(E)$, we have that it is a subalgebra of the algebra of all bounded operators on E. Noting that the limit of any norm-convergent sequence of adjointable operators $\{t_i\}$ in $\mathcal{L}(E)$ has as an adjoint, the limit of the sequence $\{t_i^*\}$ it follows that $\mathcal{L}(E)$ is closed in the operator norm. Moreover, we have that

$$\begin{aligned} ||t^*t|| &\geq \sup\{||\langle t^*tx, x\rangle|| : x \in E_1\} \\ &= \sup\{||\langle tx, tx\rangle|| : x \in E_1\} = ||t||^2 \end{aligned}$$

so $\mathcal{L}(E)$ is a C^*-algebra. Another C^*-algebra of endomorphisms on E that we'll want to consider is the algebra of compact endomorphisms, which we'll now define.

First, we need to define a finite rank endomorphism on E to be an endomorphism xy^* with $x, y \in E$ described by $xy^*(z) = x\langle y, z\rangle$. Also taking $u, v \in E$, we get that finite rank endomorphisms multiply to give finite rank endomorphisms by

$$\begin{aligned} uv^*\big(xy^*(z)\big) &= uv^*(x\langle y, z\rangle) = u\langle v, x\langle y, z\rangle\rangle \\ &= u\langle v, x\rangle\langle y, z\rangle = u\big(\langle y\langle x, v\rangle, z\rangle\big) \\ &= u\big(y\langle x, v\rangle\big)^*(z) \end{aligned}$$

and additionally that each finite rank endomorphism xy^* has the finite rank endomorphism yx^* as its adjoint because

$$\begin{aligned}\langle xy^*(z), w\rangle &= \langle x\langle y, z\rangle, w\rangle = \langle z, y\rangle\langle x, w\rangle \\ &= \langle z, y\langle x, w\rangle\rangle = \langle z, yx^*(w)\rangle.\end{aligned}$$

Consequently the closed linear span of these finite rank endomorphisms on E forms a C^*-algebra which we'll denote by $\mathcal{K}(E)$, and whose elements are called the compact endomorphisms on E.

Additionally, taking t in $\mathcal{L}(E)$, we have that $t(xy^*) = (tx)y^*$ and that $(xy^*)t = x(t^*y)^*$, which also gives us that $\mathcal{K}(E)$ is a (closed two-sided) ideal in $\mathcal{L}(E)$.

A note on the compact endomorphisms defined above (not from [11]) is that if E is a Hilbert sub$A$-module of F, and t is a compact endomorphism on E, then expressing t as a convergent series of finite rank endomorphisms $\sum x_i y_i^*$ on E, each of these endomorphisms is also finite rank on F, and consequently, t is also a compact endomorphism on F (in fact, it is exactly the endomorphism on F defined by taking t on E, and otherwise projecting down to E, so that the resulting endomorphism would be uniformly 0 on the quotient F/E). Using the quotient construction for t on F, we see that if s is a compact endomorphism on F satisfying $s \leq t$, then s must also take the value 0 on F/E, making it the extension of some compact endomorphism on E; this means that the natural extensions of $\mathcal{K}(E)$ on F form a hereditary sub-C^*-algebra of $\mathcal{K}(F)$.

Finally a more general result, useful for describing Hilbert C^*-modules as closed right ideals is the Cohen factorization given by Pedersen in [13]. To use this, we must first note that each Hilbert C^*-module is a Banach module by the construction $||x|| = ||\langle x, x\rangle||^{1/2}$ and also define the notion of an algebra A acting non-degenerately on a module E to hold when AE is dense in E. The following are taken from [13] and are presented without proof.

Theorem 2.6 (Pedersen). *Let A be a C^*-algebra acting non-degenerately on a left Banach A-module E. Then each vector ξ in E can be factored as $\xi = x\eta$ for some η in E and x in A_+ with $||x|| \leq 1$. Moreover, for each $\epsilon > 0$ we can arrange that $||\xi - \eta|| < \epsilon$.*

Theorem 2.7 (Pedersen). *Let A be a C^*-algebra acting non-degenerately on a left Banach A-module E, and let $\{\xi_n\}$ be a sequence in E such that $\sum ||\xi_n||^p < \infty$ for some $p \geq 1$ (respectively $\lim ||\xi_n|| = 0$). There is an x in A_+ with $||x|| \leq 1$, and a sequence $\{\eta_n\}$ in E with $\sum ||\eta_n||^p < \infty$ (respectively $\lim ||\eta_n|| = 0$), such that $x\eta_n = \xi_n$ for all n. Moreover, for each $\epsilon > 0$ we can arrange that $\sum ||\xi_n - \eta_n||^p \leq \epsilon$ (respectively $\sup ||\xi_n - \eta_n|| \leq \epsilon$).*

Chapter 3

The Commutative case

An obvious subject for initial study of the Cuntz semigroup would be its value on commutative C^*-algebras. Since either formulation of the Cuntz semigroup is stable under tensor products with matrix algebras, this additionaly provides us with information on the inductive limit building blocks for AH algebras.

To examine this, we'll first investigate the elements of the Cuntz semigroup arising from positive elements in the algebra $C(X)$ itself, and then look at the subsemigroup this generates in the entire Cuntz semigroup. The first observation to make is when one positive element of $C(X)$ is less than another in the semigroup.

Proposition 3.1. *Given $a, b \in C(X)$, it holds that $a \precsim b$ iff $\{x \in X; a(x) = 0\} \supseteq \{x \in X; b(x) = 0\}$ (i.e. the zero set of b is a subset of the zero set of a).*

Proof. For the forward direction, consider that the conclusion fails, i.e. that there is some $x \in X$ satisfiying $a(x) \neq b(x) = 0$. Taking $d_n(x) = (c_n^* b c_n)(x) = \overline{c_n(x)} b(x) c_n(x) = 0$ for any sequence c_n, it becomes clear that the value of the limit of any such sequence at x will also be 0, preventing equality with a, and providing the forward direction by contraposition.

For the reverse direction, take $c_n(x) = \min(\sqrt{a(x)/\max(b(x), \frac{1}{n})}, n)$. Then $(c_n^* b c_n)(x) =$

$\min(a(x)/max(b(x), \frac{1}{n}), n^2)b(x) = \min(a(x), a(x)nb(x), n^2b(x))$ where the latter values, of which the minimum is taken, get big as n does, so they will eventually exceed $a(x)$, resulting in a limit of a for $c_n^*bc_n$, which gives $a \preceq b$. □

Following directly from this, we have the following:

Lemma 3.2. *Given $a, b \in C(X)$, it holds that a and b are Cuntz equivalent iff a and b have identical zero sets (i.e. $\{x \in X; a(x) = 0\} \equiv \{x \in X; b(x) = 0\}$)*

The next obvious step here, is to look at the subsemigroup of $W(C(X))$ generated by such equivalence classes of positive elements in $C(X)$ itself (rather than those from some matrix over $C(X)$), which we'll call $U(C(X))$. The first step here is to represent the equivalence classes of positive operators with the characteristic functions of the open sets on which they are nonzero (these functions are in the classes they represent whenever they're actually in $C(X)$ – for the purposes of actually working in $C(X)$ they can be taken to fall off to 0 on a region suitably close to their boundary). It then remains to determine how semigroup addition (i.e. taking the equivalence class of direct sums of representative elements) acts on these elements.

First, it helps to ignore the equivalence classes, and just consider the direct sums. These can be represented as open sets in $X \times \mathbb{N}$ (or $\mathbb{N} \cup \{\infty\}$ if the Cuntz semigroup is defined in terms of $\mathcal{K} \otimes A$ rather than $M_\infty(A)$). In particular the pair (x, n) will be in the set representing some sum of rank-one classes, if x is nonzero in the nth summand. Ho's result from [5] (included as Lemma 2.1) then allows us to characterize $U(C(X))$ as follows:

Theorem 3.3. *The trivial subsemigroup $U(X)$ of the Cuntz semigroup, consisting of Cuntz classes generated by elements of $C(X)$, is isomorphic as a semigroup to the semigroup of lower-semicontinuous $\mathbb{N}$-valued functions (or $(\mathbb{N} \cup \{\infty\})$-valued functions, in the stable case) on X, with pointwise addition.*

Proof. First, observe that for any n, the set of x for which (x, n) is in the subset described above is an open subset of X. Now for any such subset of $X \times \mathbb{N}$, assign to x the number of distinct n for which (x, n) is in the subset. More succinctly, define the lower semicontinuous function to be the rank of the fibre on X.

It remains to show that any two subsets of $X \times \mathbb{N}$ which have the same counts on all $x \in X$ are equivalent. This is done by reducing them to a canonical form. To do this, first consider when a subset S contains points of the form $(x, n+k)$, where it doesn't contain points of the form (x, n). Letting Y_i denote the complement of the ith summand's zero set (i.e. Y_{n+k} is the set of x for which $(x, n+k)$ is in S) and Z_i the ith summand's zero set, note that $\partial Y_{n+k} \cap Z_n$ can be completed, as a path enclosing $Y_{n+k} \cap Z_n$ by taking sections through $Y_n \cap Y_{n+k}$. Because Y_{n+k} is open, $\partial Y \subseteq Z_{n+k}$, so representatives of the nth and $n+1$st summands can be chosen which are equal on this entire path. This allows us to use Lemma 2.1 to obtain an approximately unitarily equivalent (and therefore Cuntz equivalent) positive operator whose representation as a subset of $X \times \mathbb{N}$ does not contain points $(x, n+k)$ where (x, n) is not also contained (though there still may be holes at levels other than n).

Given an n, this equivalence can then be used to remove any holes (i.e. points belonging to $Z_n \cap Y_{n+k}, k > 0$) by sequentially working up through the possible values of k. Repeating this argument for all $n \in \mathbb{N}$, a canonical form C is produced where $(x, n) \in C$ exactly when (x, i) occurs in C for at least n distinct values of i, so all sums of Cuntz classes in $C(X)$ sharing these counts are equivalent.

That sums of Cuntz classes not having the same canonical form are not equivalent is a direct consequence of the fact that the rank of a product of two matrices, is always less than or equal to the lesser of the ranks of each of the matrices, so an element with greater fibre rank at any given point cannot be majorized by an element with lesser fibre rank, in the Cuntz semigroup, and therefore cannot be equivalent. This also establishes that the most complete apparent order relation for the characterization of U as $\mathbb{N} \cup \{\infty\}$-valued functions is, in fact, the Cuntz order relation $\square$

Now given the characterization of this subsemigroup, it doesn't seem unreasonable that $U(C(X))$ contains all the information that X does ad therefore, uniquely determines X (and therefore $C(X)$) up to homeomorphism (or isomorphism). Of course, since U is only defined on commutative C^*-algebras, it seems most productive to examine how it might be possible to isolate U within W, in order to use it to classify commutative (or perhaps even homogeneous) C^*-algebras. That said, it is still necessary to first determine X from $U(C(X))$. To do this, it is helpful to isolate the elements of rank one (i.e. the characteristic functions of open sets), and to do so without recourse to structure not inherent to the (ordered) semigroup.

Definition An element of an ordered semigroup will be called large either if it can be expressed as a sum of n copies ($n > 1$) of some nonzero element of the semigroup, or if is greater than such a sum. The remaining elements of the semigroup will be called small.

Proposition 3.4. *An function in $U(C(X))$ (which we recall is isomorphic to the semigroup of lower semicontinuous $\mathbb{N}(\cup\{\infty\})$-valued functions) is small iff it only takes the values 0 and 1.*

Proof. For the forward direction, consider $f \in U(C(X))$ where $f(x) = n > 1$ for some $x \in X$. Then $S = \{x \in X; f(x) = n\}$ is a nonempty open set in X. Considering the characteristic function of S; $\chi_S \in U(C(X))$, the sum of n copies of χ_s is less than or equal to f, so f is large.

For the reverse direction, observe that any nonzero element of $U(C(X))$ takes a nonzero value m on some point in X. The sum of n copies of this element will be $nm > m \geq 1$ at that point, and any element of $U(C(X))$ greater than this sum will also take a value greater than 1 at that point, so every function in $U(C(X))$ taking only values from $\{0,1\}$ must be small. □

It follows from this that all the small elements of $U(C(X))$ are the characteristic functions of open sets in X. Since χ_X is the greatest such function, let's use 1_U to denote the greatest small function in $U(C(X))$.

Now when $C(X)$ is separable, X is Hausdorff, so any singleton $\{x\}$ is closed in X and, more importantly $S_x = X\backslash\{x\}$ is open. Additionally, 1_U is the only small function greater than χ_{S_X} and any small function whose only majorizing small function is 1_U is the characteristic function of S_x for some $x \in X$.

Consequently, we can construct a set $X' = \{f \in U(C(X)); f$ is small, $1_U > f$, and $g > f \Rightarrow g = 1_U\}$ with the obvious bijection from X, $\iota : x \mapsto \chi_{S_x}$. Also, since any subset of X not containing x is a subset of $X\backslash\{x\}$, any open subset V of X not containing x will satisfy $\chi_V \leq \iota(x)$. This allows a topology $\mathcal{T}$ to be defined on X' by $V \in \mathcal{T}$ iff there is a small $f \in U(C(X))$ where $x \in V \Leftrightarrow f \not\leq x$. Since f is the characteristic function of an open subset of X, V is simply the image of that open set under ι, so the open sets on X' are exactly the images under ι of the open sets on X. Thus

Theorem 3.5. *If X is T_0, then X is homeomorphic to the space X' of maximal small elements in the trivial subsemigroup $U(C(X))$ of the Cuntz semigroup $W(C(X))$ of the commutative C^*-algebra $C(X)$.*

This also gives

Corollary 3.6. *$C(X)$ is isomorphic, as a C^*-algebra, to $C(X')$, so the trivial subsemigroup classifies commutative C^*-algebras up to isomorphism.*

This gives further that if $U(C(X))$ is isomorphic to $U(C(Y))$ as an ordered semigroup, then $C(X)$ and $C(Y)$ must be themselves isomorphic.

So in the cases where $U(C(X))$ can be isolated within $W(C(X))$ using only the semigroup structure, $W(C(X))$ provides classification. If $W(C(X))$ contains no nontrivial (i.e. not within $U(C(X))$) classes of positive operators, then isolation isn't even necessary.

In particular, Pedersen and Grove already identified a set of conditions, (in Theorem 5.6 of [7]) on a topological space X under which all normal (and therefore all positive) elements in $M_\infty(C(X))$ are unitarily equivalent (and therefore Cuntz equivalent) to a diagonal element (i.e. an element of $U(C(X))$). Their theorem follows (verbatim, without proof).

Theorem 3.7 (Grove, Pedersen; 1984). *For a compact Hausdorff space X the following conditions are equivalent:*

1. *For each n and every commutative C^*-subalgebra $\mathfrak{A}$ of $C(X) \otimes M_n$, which is countably generated over the center, there is a unitary U in $C(X) \otimes M_n$ such that $U\mathfrak{A}U^*$ consists entirely of diagonal elements*

2. (a) *X is sub-Stonean;*

   (b) *dim $X \leq 2$;*

   (c) *$H^1(X_0, S_m)$ is trivial for every closed subset X_0 of X and all m;*

   (d) *$H^2(X_0, \mathbb{Z})$ is trivial for every closed subset X_0 of X.*

Unfortunately, the set of restrictions imposed by this theorem is rather strict, and met by very few commutative algebras of general interest (e.g. any metrizable space is not sub-Stonean).

Fortunately, for $U(C(X))$ to be the whole of $W(C(X))$ it is only necessary to diagonalize positive operators (rather than all normal operators), and further, Cuntz equivalence is a considerably weaker condition than unitary equivalence. In particular, when X is a CW-complex of dimension 1, then $W(C(X))$ can be shown to be equal to $U(C(X))$, however the proof of this uses machinery to be explained later in this thesis, and will come later.

Chapter 4

A Suitable category

Even with indications that commutative C^*-algebras may be classified by the Cuntz semigroup, viewed strictly as a semigroup, it still has considerable deficiencies. One such deficiency arises when considering the Cuntz semigroup of an inductive limit algebra.

As an example, consider the UHF-algebra A, which has $K_0(A) = \mathbb{Q}$. Given any $x \in \mathbb{R}^+$, we can construct an increasing (in the C^*-algebra) sequence of projections p_n such that $\lim_{n\to\infty}[p_n] = x$ (where $[p_n]$ is the Murray-von Neumann equivalence class of p_n considered as an element of $\mathbb{Q} \subseteq \mathbb{R}$). Then

$$\sum_{n=1}^{\infty} p_n/2^n$$

is a positive element not (Cuntz) equivalent to any projection in A (even if x is rational—consider the closed right ideals generated by the given sum versus those generated by a single projection with trace x; the sum's ideal can be seen as countably generated by all the projections smaller than the corresponding single projection and therefore not isomorphic to the single projection's ideal, so the operators can't be Cuntz equivalent).

A being AF, we also get that all positive elements can be approximated by such sequences of projections, so we have that $W(A) = (\mathbb{Q}+\mathbb{R})^+$ (this notation being used to describe the disjoint union of the strictly positive real numbers, and the positive rational numbers with the usual addition on each of these subsets, and addition of one element of each yielding the expected numerical sum in the $\mathbb{R}$ portion). This is uncountable, but the Cuntz semigroups of all the building blocks are countable (simply being $\mathbb{N}$, since the building blocks are matrix algebras), it is apparent that algebraic structure alone cannot produce an inductive limit semigroup, for the building block semigroups, which is isomorphic to the Cuntz semigroup of the inductive limit algebra.

Drawing on the analogy to $\mathbb{Q}$ and $\mathbb{R}$ suggested by the example of A however, there appears to be promise in topological properties of the Cuntz semigroup. Even more promising is that such topological properties can be identified because the Cuntz semigroup comes equipped with an order relation (the preorder relation used in its definition, after the quotient is taken by equivalence), and that this order relation compares more elements than the usual semigroup ordering (i.e. $x \leq y$ iff there is a z with $y = x + z$; indeed, it is shown in [14] that in a C^*-algebra with stable rank one, the set of Cuntz classes majorizing a class x agree iff x is the equivalence class of a projection).

Before listing the properties that define the desired category, it is worth noting again that the material in this chapter, and the next two, is joint work, published with George Elliott and Cristian Ivanescu in [2].

4.1 Defining the category $\mathcal{C}$

In order to refine the order properties so that, e.g. the limit rationals in $\mathbb{R}^+$ from the example above are kept distinct from the rational images of building block elements (in $\mathbb{Q}^+$), a notion of compact containment needs to be introduced.

Definition An element of x of an ordered semigroup S is said to be compactly contained in y (or "way less than y"; denoted $x \ll y$) iff whenever $y_1 \leq y_2 \leq \cdots$ is an increasing sequence with supremum greater than or equal to y, there is an n for which $x \leq y_n$.

Remark It is worth noting here first that $x \ll y$ implies $x \leq y$, as well as that either of $x \leq y \ll z$ or $x \ll y \leq z$ imply $x \ll z$. For the first implication, any sequence $\{y_i\}$ with y as a supremum will have, for some j, $x \leq y_j \leq y$. For the second, given a series $\{z_i\}$ increasing to a supremum of at least z, $y \ll z$ gives a j so that $x \leq y \leq z_j$, so the same j serves to demonstrate that $x \ll z$. For the final implication, any increasing sequence having a supremum of at least z, has a supremum of at least y, and so satisfies the condition of eventually exceeding x that establishes $x \ll z$. These implications are used quite often in the rest of this chapter.

Definition We also say that a sequence $\{y_n\}$ is rapidly increasing when $y_1 \ll y_2 \ll \cdots$

Definition The category $\mathcal{C}$ consists of objects which are semigroups with zero, an order relation compatible with addition (though not necessarily arising from it), and the following additional properties:

1. every increasing sequence (or countable, upward directed set) has a supremum;

2. for every element y, the set of elements compactly contained in y is upward directed, and contains a rapidly increasing sequence, of which y is the supremum;

3. the operation of taking suprema of countable, upward directed sets is compatible with addition (i.e. for two such sets S_1 and S_2, $\sup(S_1 + S_2) = \sup S_1 + \sup S_2$)

4. the relation of compact containment is compatible with addition (i.e. $x_1 \ll y_1$ and $x_2 \ll y_2$ implies $x_1 + x_2 \ll y_1 + y_2$).

   and arrows which are order-preserving semigroup homomorphisms between these objects.

4.2 Membership of candidate inductive limits in $\mathcal{C}$

The category having now being defined, recall that the motivation for this category was that the operation of taking the Cuntz semigroup from a C^*-algebra to it be a functor which preserves inductive limits. In order for this to make sense, it is necessary to have a well-defined notion of inductive limits within $\mathcal{C}$.

To do this, first take, for a sequence

$$S_1 \to S_2 \to S_3 \to \cdots$$

in $\mathcal{C}$, a preliminary semigroup S' of eventually increasing sequences $\{s_1, s_2, s_3, \ldots\}$ (where $s_n \in S_n$ for every n, and eventually increasing is taken to mean that, there is a k such that for any $i \geq k$, the image of s_i in S_{i+1} is less than or equal to s_{i+1}), and with addition defined by

$$\{s_i\} + \{t_i\} = \{s_i + t_i\}$$

.

First, we'll want to define the increasing part of an eventually increasing sequence $\{s_i\}$ as $\{s_i\}_{i \geq k}$ where k is the smallest number for which the image of s_i in S_{i+1} is less than or equal to s_{i+1} for any $i \geq k$.

Now we can assign to this semigroup the preorder relation $\{s_i\} \precsim \{t_i\}$ when, for any $s \in S_i$ with $s \ll s_i$, when s_i is in the increasing part of $\{s_i\}$, there is a $j > i$ so that t_j is in the increasing part of $\{t_i\}$ and the image t, of s in S_j satisfies $t \ll t_j$. Now take the candidate semigroup S to be the quotient of S' by the equivalence relation produced by this preorder (i.e. $\{s_i\} \precsim \{t_i\}$ and $\{t_i\} \preceq \{s_i\}$). This naturally requires some checking.

Proposition 4.1. *The relation: $\{s_i\} \precsim \{t_i\}$ when, for any $s \in S_i$ with $s \ll s_i$ where s_i is in the increasing part of $\{s_i\}$, there is a j in the increasing part of $\{t_i\}$, $j > i$ so* *that the image t of s in S_j satisfies $t \ll t_j$; is a preorder relation (i.e. it is reflexive and transitive).*

Proof. That $\{s_i\} \precsim \{s_i\}$ follows directly from taking $j = i$. For transitivity, take $\{s_i\} \precsim \{t_i\}$ and $\{t_i\} \precsim \{r_i\}$. Taking $s \in S_i$ with $s \ll s_i$ and s_i in the increasing part of $\{s_i\}$, we get a j in the increasing part of $\{t_i\}$ so that the image t of s in S_j satisfies $t \ll t_j$. Further, we get that the image r of t in S_k (again, r_k in the increasing part of $\{r_i\}$) satisfies $r \ll r_k$. Since r is also the image of s, then $\{s_i\} \precsim \{r_i\}$, so the relation is transitive and a preorder. □

Now it also helps, in order to have an ordered semigroup, that addition be well-defined, and compatible with the order structure. In particular, note that if $s_i \leq s_{i+1}$ and $t_i \leq t_{i+1}$, then compatbility of addition with the ordering in S_{i+1} gives $s_i + t_i \leq s_{i+1} + t_{i+1}$, so it remains to check that

Proposition 4.2. *If $\{s_i\}, \{s'_i\}, \{t_i\}$, and $\{t'_i\}$ are eventually increasing sequences in $\{S_i\}$ as described above, and if $\{s_i\} \precsim \{s'_i\}$ and $\{t_i\} \precsim \{t'_i\}$, then $\{s_i + t_i\} \precsim \{s'_i + t'_i\}$.*

Proof. Take $s \in S_i$ (again, in the increasing part) so that $s \ll s_i + t_i$, in order to eventually show that there is a j (in the increasing part) so that the image of s in S_j (which will also be called s from now on) satisfies $s \ll s'_j + t'_j$.

Now because of the second property assigned to objects in $\mathcal{C}$, sequences $\{s_i^n\}$ and $\{t_i^n\}$, with suprema s_i and t_i respectively, can be taken in S_i, satisfying

$$s_i^1 \leq s_i^2 \leq \cdots \ll s_i \text{ and } t_i^1 \leq t_i^2 \leq \cdots \ll t_i$$

(where the requirement $s_i^n \ll s_i$ and $t_i^n \ll t_i$ follows from the ability to take rapidly increasing sequences). Additionally, compatibility between addition, the order relation, and suprema in $\mathcal{C}u$ establishes that

$$s_i^1 + t_i^1 \le s_i^2 + t_i^2 \le \cdots \le s_i + t_i \text{ with } s_i + t_i = \sup_n \{s_i^n + t_i^n\}.$$

So since $s \ll s_i + t_i$, then there is an n so that $s \le s_i^n + t_i^n$. Also, because we have $s_i^n \ll s_i$, $t_i^n \ll t_i, \{s_k\} \le \{s'_k\}$, and $\{t_k\} \le \{t'_k\}$, the definition of comparison between sequences gives the existence of a j (in the increasing parts of both $\{s'_i\}$ and $\{t'_i\}$) satisfying both $s_i^n \ll s'_j$ and $t_i^n \ll t'_j$. Thus compatibility of addition and order gives us

$$s \le s_i^n + t_i^n \le s'_j + t'_j$$

Which establishes that $\{s_i + t_i\} \precsim \{s'_i + t'_i\}$. □

Having established S' to be a preordered semigroup, whose preorder is compatible with addition, the quotient S is a well-defined ordered semigroup, whose inclusion as an object in $\mathcal{C}$ can now be checked. To do this, we start with the following:

Proposition 4.3. *Every eventually increasing sequence $\{s_1, s_2, \ldots\}$ with $s_i \in S_i$ is equivalent to an eventually rapidly increasing sequence.*

Proof. For each i in the increasing part, we have a rapidly increasing sequence $\{s_i^n\}$ in S_i, which has s_i as a supremum. Because these sequences are rapidly increasing, a subsequence of $\{s_{i+1}^n\}$ can be chosen so that the nth term of this sequence is always greater than or equal to s_i^n. Starting this process with the $\{s_{k+1}^n\}$ (where s_k is the first term of the rapidly increasing part of $\{s_i\}$ and continuing with each subsequent sequence, yields a sequence of sequences which is increasing over i and rapidly increasing over n, so consider $\{s_i^n\}$ to have been chosen to meet these conditions. Constructing a Cantor diagonal sequence $\{t_i\}$ by taking $\{t_i\} = \{s_i^i\}$ (for i in the increasing part of $\{s_i\}$, and arbitrary values for lower values of i) so that $s_i^i \ll s_i^{i+1} \le s_{i+1}^{i+1}$ provides that $t_t \ll t_{i+1}$, i.e. that $\{t_i\}$ is eventually rapidly increasing.

To get that $\{s_i\} \precsim \{t_i\}$, first take s and i (in the increasing part; note that choosing suitably large elements for the arbitrary part of $\{t_i\}$, gives the same initial index for the increasing parts of both sequences) so that $s \ll s_i$, that is, so there is a j satisfying $s \leq s_i^j$. Taking $k = \max(i, j) + 1$, we get

$$s \leq s_i^j \ll s_i^k \leq s_k^k = t_k$$

and so with $s \ll t_k$, we have $\{s_i\} \precsim \{t_i\}$.

For $\{t_i\} \precsim \{s_i\}$, taking s and i (again, i in the increasing part) so that $s \ll t_i$, it need only be noted that $t_i = s_i^i$, that is the ith term of a sequence rapidly increasing to a supremum of s_i, so $s \ll t_i \ll s_i$, giving $\{t_i\} \precsim \{s_i\}$, which further gives that the eventually rapidly increasing sequence $\{t_i\}$ is equivalent to the eventually increasing sequence $\{s_i\}$. □

Proposition 4.4. *If $\{s_i\} \preceq \{t_i\}$, and $\{s_i\}$ is equivalent to a eventually rapidly increasing sequence $\{s_i'\}$ with rapidly increasing part $\{s_i'\}_{i \geq k}$, then $\{t_i\}$ is equivalent to an eventually rapidly increasing sequence $\{t_i'\}$ which is rapidly increasing on $\{t_i'\}_{i \geq k}$.*

Proof. Noting first that Proposition 4.3 provides an eventually rapidly increasing sequence equivalent to $\{t_i\}$, consider the sequence provided to be eventually rapidly increasing, with rapidly increasing part $\{t_i\}_{i \geq n}$. $\{t_i\}$ already being rapidly increasing for $i \geq k$ if $n \leq k$, we need only consider the case where $n > k$. In this case, note that $s_k \ll s_{k+1}$ and $\{s_i\} \preceq \{t_i\}$, so there is a $j \geq n$ with $s_k \ll t_j$. Now recall that in S_k, there is a rapidly increasing sequence

$$s_k^1 \ll s_k^2 \ll s_k^3 \ll \cdots \ll s_k \ll t_j$$

so taking the images first $j - k$ terms of this sequence in S_k through S_{j-1}, we can get the eventually rapidly increasing sequence

$$s_1, s_2, ...s_k - 1, s_k^1 \ll s_k^2 \ll \cdots \ll s_k^{j-k} \ll t_j \ll t_{j+1} \ll \cdots$$

which is rapidly increasing for $i \geq k$, and is equivalent to $\{t_i\}$, as desired. $\square$

Lemma 4.5. *The ordered semigroup S is an object in $\mathcal{C}$, that is, it is an ordered semigroup, with zero, whose order relation is compatible with addition, has suprema of countable, upward directed sets, has upward directedness of the set of elements contained in any given element (and a rapidly increasing sequence therein, whose supremum is the given upper bound), and has compatibility between addition, taking suprema, and the $\ll$ ordering.*

Proof. To get a zero element for S, simply take the equivalence class of the sequence $\{0, 0, \cdots\}$, which is a zero element of S'. That the order relation is compatible with addition also follows from the work done with the preorder on S'.

To get that each increasing sequence $s^1 \leq s^2 \leq \cdots$ within S has a supremum, take eventually rapidly increasing representatives $\{s^1_n\} \precsim \{s^2_n\} \precsim \cdots$ $(s^i_n \in S_n)$ for each of $s^i \in S$, and use Proposition 4.4 to get that they're all rapidly increasing on the same set of indices $(i \geq k)$. Repeating the subsequence construction in Proposition 4.3 (but taking the subsequences 1 entry further along each of the original sequences to ensure compact containment), we can get sequences satisfying

$$s^1_i \ll s^2_i \ll \cdots$$

for $i \geq k$. This also gives rapid increasingness of the diagonal sequence

$$s^1_{k+1} \ll s^2_{k+2} \ll \cdots$$

the class of which, we'll call s, and take as our candidate for the supremum of $s_1 \leq s_2 \leq \cdots$.

To establish that s is an upper bound for $\{s^i\}$, simply note that for any i and n, $s^i_n \leq s^m_{k+m}$ for $m = \max(i, n)$, so $s_i \leq s$. Then considering $t = (t_1, t_2, \ldots) \succsim s^i$ for every i, taking $r \ll s^i_i$ (the term of $\{s^i_i\}$, in S_i with r), that s^i_i is a term in s^i, and $s^i \precsim t$ gives the existence of a j in the increasing part of $\{t_i\}$ so that $r \leq t_j$, which means that $\{s_i^i\} \precsim t$, so s is the supremum for $\{s^i\}$.

Next we need that every element of S is the supremum of a rapidly increasing sequence (in S). Representing $s \in S$ by the eventually rapidly increasing sequence $\{s_i\}$ with $s_i \in S_i$ and rapid increasingness on $i \geq k$, we construct a candidate sequence, from the equivalence classes represented by

$$\{s_k, s_k, s_k, \ldots\}, \{s_k, s_{k+1}, s_{k+1}, \ldots\}, \{s_k, s_{k+1}, s_{k+2}, s_{k+3}, \ldots\}$$

To get rapid increasingness, it is necessary to check that for any $l > k$,

$$s^l = \{s_k, s_{k+1}, \ldots, s_{l-1}, s_l, s_l, \ldots\} \ll \{s_k, s_{k+1}, \ldots, s_l, s_{l+1}, s_{l+1}, \ldots\} = s^{l+1}$$

For this, first take a sequence (in S)

$$t^1 \leq t^2 \leq \cdots \leq t$$

(where t is the supremum of the sequence satisfying $s^{l+1} \leq t$) and, towards establishing that there is a j with $s^l \leq t^j$, consider $p \ll s_l$ (in S_l). Then $p \ll s_{l+1}$ and since $s^{l+1} \leq t$, in any sequence $\{t_i\}$ equivalent to t, there is a j satisfying $p \ll t_j$. Constructing the supremum of $\{t^i\}$ from diagonal entries of subseqences (as was done to establish the existence of suprema), then the t_j just described (as t is the supremum of t^i, it must be equivalent to the diagonal sequence) is a term from t^j, and $p \ll t_j$, so $s^l \leq t^j$. Consequently, $s^l \ll s^{l+1}$, so the chosen sequence is rapidly increasing, leaving it to be shown that its supremum is s.

To do this, take $t \in S$ so that $s^i \leq t$ (for all i, continuing that $s^i = \{s_k, s_{k+1}, \ldots, s_i, s_i, \ldots\}$, and choose an eventually increasing sequence $\{t_i\}$ representing t (again, $t_i \in S_i$). Because $s_i \ll s_{i+1}$, and $s^{i+1} \leq t$, there is a j in the increating part of t satisfying $s_i \ll t_j$, so $s = \{s_1, s_2, s_3, \ldots\} \preceq \{t_1, t_2, t_3, \ldots\}$ so $s \preceq t$ in S, and we have our rapidly increasing sequence with supremum s.

Now it remains to test the compatibility of $\leq$, $\ll$, and suprema with addition. For suprema, consider that any $s \in S$ can be represented as the supremum of an eventually rapidly increasing sequence $\{s_1, s_2, \ldots\}$ with s_i in S_i, so taking $\{s^i\}$ to be an increasing sequence in S with supremum s, we get the eventually rapidly increasing sequence $\{s_i\}$ with $\sup s_i = \sup s^i$ and $s_i \leq s^i$ for every i. So for sequences $\{s^i\}$ and $\{t^i\}$ in S, with suprema s and t, take $\{s_i\}$ and $\{t_i\}$ to be eventually rapidly increasing representative sequences as above. By definition of addition on S, the sequence $\{s_1+t_1, s_2+t_2, \ldots\}$ is a representative sequence for $s+t$, and consequently the supremum for $\{s_i+t_i\}$ (recall the argument used to construct eventually rapidly increasing sequences). Noting also that $s_i \leq s^i$ and $t_i \leq t^i$ for all i in the rapidly increasing portion of both sequences, we have $s_i + t_i \leq s^i + t^i$, so:

$$s + t = \sup\{s_i + t_i\} \leq \sup\{s^i + t^i\} \leq \sup s^i + \sup t^i = s + t$$

which gives equality of all entries in that sequence, and proves compatbility of suprema with addition by giving

$$\sup\{s^i + t^i\} = \sup s^i + \sup t^i$$

This is now useful for proving the compatibility of $\leq$ with addition; to do this, given $s^1 \leq t^1$ and $s^2 \leq t^2$ in S, choose eventually rapidly increasing sequences $\{s_i^1\}$, $\{t_i^1\}$, $\{s_i^2\}$, and $\{t_i^2\}$, with $s_i^1, t_i^1, s_i^2, t_i^2 \in S_i$. Replacing $\{s_i^2\}$ and $\{t_i^2\}$ with subsequences so that by the shared increasing part, $s_i^1 \leq s_i^2$, and $t_i^1 \leq t_i^2$ for all i (which can be done because $\{s_i^1\} \precsim \{s_i^2\}$ and $\{t_i^1\} \precsim \{t_i^2\}$, and the sequences are eventually rapidly increasing — because $s_i^1 \ll s_{i+1}^1$, and $s^1 \leq s^2$, there is a j in the rapidly increasing part so that

$s_i^1 \ll s_j^2$, and likewise for the ts), we get:

$$\begin{aligned} s^1 + t^1 &= \sup s_i^1 + \sup t_i^1 \\ &= \sup\{s_i^1 + t_i^1\} \\ &\leq \sup\{s_i^2 + t_i^2\} \\ &\leq \sup s_i^2 + \sup t_i^2 \\ &= s^2 + t^2 \end{aligned}$$

i.e. $s^1 + t^1 \leq s^2 + t^2$, as desired.

Now for the compatibility between addition and $\ll$, simply take $s^1 \ll s^2$ and $t^1 \ll t^2$ in S with eventually rapidly increasing representative sequences $\{s_i^2\}$ and $\{t_i^2\}$ ($s_i^2, t_i^2 \in S_i$) for s^2 and t^2. Because $s^1 \ll s^2$, considering only the increasing part, we get that eventually $s^1 \leq s_i^2$, and likewise $t^1 \leq t_i^2$, so

$$s^1 + t^1 \leq s_i^2 + t_i^2 \ll s_{i+1}^2 + t_{i+1}^2 \leq s^2 + t^2$$

which yields the desired $s^1 + t^1 \ll s^2 + t^2$.

So all the conditions being met, the quotient semigroup S, taken from the quotient of the preordered semigroup S' of sequences in

$$S_1 \to S_2 \to S_3 \to \cdots$$

is an ordered semigroup in the category $\mathcal{C}u$.

□

4.3 Universal property of the candidate inductive limits

The semigroup S having been demonstrated to be an ordered semigroup in $\mathcal{C}$, it still remains to show that it is the inductive limit of the sequence $S_1 \to S_2 \to \cdots$. In particular, we need that for every ordered semigroup T in $\mathcal{C}$, and every sequence of maps $S_i \to T$ (compatible in the sense that the map $S_i \to T$ is equal to the composition of the maps $S_i \to S_{i+1}$ and $S_{i+1} \to T$), there is a unique compatible map $S \to T$ making the following diagram commutative:

$$\begin{array}{ccccccc} S_1 & \to & S_2 & \to & \cdots & \to & S \\ & \searrow & & \searrow & & & \downarrow! \\ & & & & & & T \end{array}$$

Proposition 4.6. *Given a sequence $S_1 \to S_2 \to \cdots$ in $\mathcal{C}$, there are maps $S_i \to S$ which are compatible with the sequence maps in that they form a commutative diagram. Further, assuming the existence of $T \in \mathcal{C}$ with similarly compatible maps, there is a well-defined, map ϕ from S to T, which is compatible with both sets of maps (i.e. $S_i \to S \to T$ agrees with $S_i \to T$).*

Proof. This first requires the existence of maps for each i, $S_i \to S$, compatible with the maps $S_i \to S_{i+1}$ in the sense described for the maps to T. Considering the map which takes $s \in S_i$ to the class of the sequence $(a_1, a_2, \ldots, a_{i-1}, s, s, \ldots) \in S$, (where the a_n can be any elements from the appropriate semigroups) compatibility follows immediately from the preservation of order (and consequently of compact containment) by morphisms in $\mathcal{C}$, and the definition of the preorder relation on S'.

Given T with the compatible maps from S_i, a map $\phi : S \to T$ can be constructed, first by considering only eventually rapidly increasing representative sequences for elements of S. In particular, take the sequence $\{s_1, s_2, \ldots\}$, $s_i \in S_i$ to represent $s \in S$ (with $s_k \ll s_{k+1} \ll \cdots$). Now consider mapping s to the supremum of the sequence constructed from the image of the rapidly increasing part of $\{s_i\}$. To see that this mapping is well defined, consider a second eventually rapdily increasing sequence $\{s'_l, s'_{l+1}, \ldots\}$ also with supremum s. Because for $i \geq k$, $s_i \ll s_{i+1} \leq s$ and $\{s'_i\}$ is an eventually increasing sequence with a supremum of at least s, there is a j in the increasing part of $\{s'_i\}$ so that $s'_j \geq s_i$; similarly, each s'_i also admits a j with $s_j \geq s'_i$ (again in increasing parts). Because the maps from $S_i \to T$ are $\mathcal{C}$ morphisms, they preserve order, so the intertwining of $\{s_i\}$ and $\{s'_i\}$ gives the same supremum for the two sequences, making the map from S to T well defined.

It now remains to show the compatibility between $S_k \to T$ and $S_k \to S \to T$. For $s \in S_k$, the image chosen is the class of $\{s, s, \ldots\}$, but this is not necessarily rapidly increasing. Taking $\{r_1, r_2, \ldots\}$ to be a rapidly increasing sequence in S_k with supremum S, we get the sequence $\{0, 0, \ldots, r_i, r_{i+1}, \ldots\}$ (i.e. r_j in th jth place for $j \geq k$ and 0 for all other entries), is an eventually rapidly increasing sequence, with rapidly increasing part $\{r_i\}_{i \geq k}$, and whose supremum (for the rapidly increasing part) in S_i is s (equivalence follows from all entries in the rapdily increasing part of $\{r_i\}$ being way less than s, so anything way less than one of them would also be way less than s; and because the sequence increases to s, so any $q \ll s$ would be less than one of its elements, and way less than the next). Now the image in T of the rapdily increasing portion of this representative of s is just the image of the rapidly increasing sequence $\{r_i\}_{i \geq k}$, which has supremum s in S_i for $i \geq k$ and also in T, because the $\mathcal{C}$ morphisms preserve suprema of increasing sequences, and the map from S_k to each of these semigroups is just such a morphism. □

Now having the desired compatibility of the maps, to get S as an inductive limit for $S_1 \to S_2 \to \cdots$ it remains only to show that ϕ is in fact a $\mathcal{C}$ morphism from S to T, i.e. that it preserves addition, preserves the order relation, preserves suprema of increasing sequences, and preserves the $\ll$ relation.

Theorem 4.7 (Coward, Elliott, Ivanescu). *The map ϕ is a $\mathcal{C}$ homomorphism. Consequently, S is the inductive limit of $S_1 \to S_2 \to \cdots$, and therefore, inductive limits exist in $\mathcal{C}$.*

Proof. For addition, consider two eventually rapidly increasing sequences $\{r_1, r_2, \ldots\}$

and $\{s_1, s_2, \ldots\}$, and let k be the smallest value by which both of these sequences and their sum have become rapidly increasing. For each $i \geq k$, the image in T of the sum $r_i + s_i$ from S_i, is the sum of the images of r_i and s_i, since the map $S_i \rightarrow T$ is a $\mathcal{C}$ morphism. Also, being such a morphism, it preserves suprema of increasing sequences so $\phi(s_i + t_i) = \phi(s_i) + \phi(t_i)$, so addition is preserved.

To check preservation of suprema by ϕ, given an increasing sequence $\{s^1, s^2, \ldots\}$ of elements of S, choose eventually rapidly increasing representatives $\{s_i^n\}$ for each s^n, as in the proof of the existence of suprema in S, so that there is a k such that $i \geq k+1$ gives $s_i^n \ll s_i^{n+1}$ as well as $s_i^n \ll s_{i+1}^n$, and take the diagonal sequence $\{s_{k+i}^i\}$ as a rapidly increasing sequence which represents $\sup s^i$. Because for any s_i^n with $i \geq k+1$, $m = \max(k-i, n)$ gives $s_{k+m}^m \geq s_i^n$, the $\sup\{s_{k+i}^i\} \geq s^n$ for any n, so the supremum in T of the images of the diagonal elements will be the supremum of the image of the sequence $\{s^n\}$.

Preservation of order by ϕ follows from the argument used to show that the map is well defined, so it remains only to show that ϕ also preserves the $\ll$ relation.

Take $\{r_1, r_2, \ldots\}$ and $\{s_1, s_2, \ldots\}$ to be eventually rapidly increasing sequences with $r_i, s_i \in S_i$, so that $\{r_i\} \ll \{s_i\}$. Taking $r = \sup r_i$ and $s = \sup s_i$ (in S), note that $r \ll s$ gives the existence of some j in the rapidly increasing part of $\{s_i\}$ so that $r \leq s_j$. Because ϕ already preserves order, we also have $\phi(r) \leq \phi(s_j)$ (where it has already been established by compatibility that $\phi(s_j)$ is also the image of s_j under the map $S_j \rightarrow T$). Considering also that the map $S_{j+1} \rightarrow T$ is a $\mathcal{C}$ morphism preserving $\ll$, and particularly $s_j \ll s_{j+1}$, and that the preservation of suprema by ϕ gives $s_{j+1} \leq s$, we get, in T

$$r \leq s_j \ll s_{j+1} \leq s$$

so $r \ll s$ as desired, ϕ is a $\mathcal{C}$ morphism, and

$$\varinjlim S_i = S$$

□

It should be noted that the semigroups in $\mathcal{C}u$ needn't be positive (i.e. have 0 as their minimal element), though the Cuntz semigroup of any C^*-algebra is positive. This will be discussed further after it has been verified that the Cuntz semigroup can actually be considered as a functor into it.

Chapter 5

A Strikingly similar functor

Examining the category $\mathcal{C}$ in light of the Cuntz semigroup as already defined, the question becomes how to prove that the given equivalence classes of positive elements, with their order relation and addition, constitute an object in $\mathcal{C}$. Indeed, it is even problematic to find suprema for all increasing sequences in the case of stable algebras (in unital algebras, with the positive elements taken from $M_\infty(A)$, it's even worse: the increasing sequence of direct sums of n copies of 1_A already has no supremum!). Even just finding the supremum of Cuntz classes of two arbitrary positive elements seem unmanageable, let alone an entire countable set.

Consequently, it makes sense to consider an alternate formulation for the Cuntz semigroup, with which it is easier to demonstrate the desired properties, and demonstrate that it is (close enough to) the established semigroup.

5.1 Defining the map

In order to find such a formulation, it helps to draw an analogy to the Murray-von Neumann semigroup. Particularly, that the Murray-von Neumann semigroup of equivalence classes of projections in $M_\infty(A)$ is isomorphic to the semigroup of algebraically finitely generated projective Hilbert C^*-modules over A. Considering that the isomorphism between the two formulations of the Murray-von Neumann semigroup, maps a class of projections to the class of modules generated by those projections, the Cuntz semigroup ought to map equivalence classes of positive elements to some sort of equivalence classes of Hilbert C^*-modules generated by those elements.

The initial consideration, given this analogy, would be to take the isomorphism classes of all algebraically finitely generated Hilbert A-modules (where a Hilbert A-module is taken to mean a Hilbert C^*-module over A). This however, runs into a few immediate problems. The first of these problems is that an increasing sequence of such modules, each with more generators than the last, could quite easily fail to have a finitely generated supremum (consider increasing classes of projections on the algebra $\mathcal{K}$ of compact operators). The second of these problems is that while the Murray-von Neumann equivalence relation on projections is defined directly enough that it could admit distinct projections p and q with $p \precsim q$, $q \precsim p$, and $p \nsim q$ (and in fact does in the Cuntz algebras $\mathcal{O}_n, n \geq 2$), the definition of Cuntz equivalence in terms of the preorder relation, requires that if $a \precsim b$ and $b \precsim a$, then $a \sim b$.

To solve the first of these problems, we can consider countably generated (rather than algebraically finitely generated) Hilbert A-modules, and for the second we could consider the following notion of compact containment for Hilbert A-modules; that closed submodule E of the Hilbert A-module F is compactly contained in F (denoted $E \subset\subset F$) iff there is a compact, self-adjoint endomorphism of F which acts as the identity upon E. The preorder relation $E \precsim F$ is then defined to hold iff every compactly contained submodule of E is isomorphic to a compactly contained submodule of F (which is pretty obviously a preorder, following from taking the submodules themselves, and from composing the isomorphisms to get reflexivity and transitivity respectively).

Before passing to the space of equivalence classes under the relation induced by thie preorder, a few checks are in order to ensure that this quotient is in fact a reasonably well-behaved semigroup (in that if E is a submodule of F, then their equivalence classes $[E]$ and $[F]$ satisfy $[E] \leq [F]$).

Proposition 5.1. *If E and F are Hilbert C^*-modules over a C^*-algebra A, and $E \subseteq F$, then $E \precsim F$ (and consequently their Cuntz equivalence classes $[E]$ and $[F]$ satisfy $[E] \leq [F]$).*

Proof. Taking E_0 to be compactly contained in E, we have $E_0 \subseteq E \subseteq F$, and a compact endomorphism on E which acts as the identity upon E_0; considering the endomorphism on F which is the identity on E, and otherwise projects on to E, so that it is 0 on the quotient F/E, and composing it with the compact endomorphism on E that gives $E_0 \subset\subset F$, we get a compact endomorphism on F, which acts as the identity on E_0, so $E_0 \subset\subset F$, giving $E \precsim F$. □

The good behaviour of the preorder relation with respect to inclusion, having been established, it remains to show that the quotient semigroup is in fact a semigroup, and further, is an object in $\mathcal{C}$. This requires some preliminaries, where the semigroup is just considered as an ordered set. In addition, a (temporary) substitution for the compact containment order needs to be made.

Definition We will say that $[E]$ is concretely compactly contained in $[F]$, denoted $[E] \subset\subset [F]$ iff there is an A-module F' satisfying $[E] \leq [F']$ and $F' \subset\subset F$.

This notion of compact containment will be used on the ordered set of equivalence classes of Hilbert A-modules instead of the order-theoretic notion (at least until they are proven to be equivalent). The first such use of this relation will be on the the following object:

Definition We will denote by $\mathcal{C}u(A)$ the ordered set (which we will later show to be a semigroup) of equivalence classes of countably generated Hilbert A-modules. We will also (eventually) call it the stable Cuntz semigroup, or the module form of the Cuntz semigroup.

And this object first appears in the following result:

Proposition 5.2. *The concretely rapidly increasing sequence $[E_1] \subset\subset [E_2] \subset\subset \cdots$ of equivalence classes of countably generated Hilbert A-modules has a supremum in the ordered set, $\mathcal{C}u(A)$ of equivalence classes of countably generated Hilbert A-modules.*

Proof. The concrete rapid increasingness of $\{E_n\}$ gives, by definition of concrete compact containment, a sequence of countably generated Hilbert A-modules: $E_2', E_3', \ldots$ satisfying $E_n' \subset\subset E_n$ for every $n \geq 2$, and $[E_n] \leq [E_{n+1}']$ for every $n \geq 1$. Noting that $E_n' \subset\subset E_n$ implies $E_n' \subseteq E_n$, giving $[E_n'] \leq [E_n]$, we get that the concretely rapidly increasing sequence $\{[E_n]\}$ is in fact an increasing sequence (and so this proposition is actually a special case of the property that increasing sequences in $\mathcal{C}$ converge to a supremum), and further, that we have the intertwining sequence

$$[E_1] \leq [E_2'] \leq [E_2] \leq [E_3'] \leq [E_3] \leq \cdots$$

which gives that $\{[E_n]\}$ and $\{[E_n']\}$ must have the same supremum.

Noting that $E_n' \subset\subset E_n$, the definition of the order $[E_n] \leq [E_{n+1}']$ gives that E_n' must be isomorphic to a compactly contained submodule of E_{n+1}' (for all $n \geq 2$). Consequently, we have a sequence of isometric A-module maps which preserve the A-valued inner product:

$$E_2' \to E_3' \to E_4' \to \cdots$$

where the image each module is compactly contained in the next. This introduces the possibility that the inductive limit $E = \lim_{\to} E_n$ provides the supremum $[E]$ for the sequences. Transitivity of compact containment, and isometry of the inclusion maps already give that $[E]$ is an upper bound for $\{[E'_n]\}$ (and therefore $\{[E_n]\}$), so it remains to show that if $[E_n] \leq [F]$ for some countably generated Hilbert A-module F and every n, then $[E] \leq [F]$.

Choosing such an F, take G to be a countably generated Hilbert A-module such that $G \subset\subset E$. The compact containment gives that there must be a G' with $G \cong G' \subseteq E$ and a compact self-adjoint endomorphism a on E which acts as the identity on G'. Noting that a^2 is positive (and still acts as the identity on G'), we can use functional calculus to take b to be a function on a so that $(b-\epsilon)_+$ is the identity on G'. Using functional calculus, we can also construct another compact self-adjoint endomorphism c, with $cb = bc = b$. Noting that the C^*-algebra of compact endomorphisms on E can be expressed as the inductive limit of the C^*-algebras of compact endomorphisms on $E_n \subseteq E$, we can construct a sequence $\{c_n\}$ of compact endomorphisms on each of E'_n, which converges to c. This gives that $c_n b c_n$ converges to $cbc = b$, so for $\epsilon > 0$, we have an n with $||c_n b c_n - b|| < \epsilon$. Then by Lemma 2.4, there exists a compact endomorphism d so that $dc_n b c_n d^* = (b-\epsilon)_+$. Noting the compact endomorphisms on any of the building block modules constitute a hereditary subalgebra of the C^*-algebra of compact endomorphisms on E, then $c_n b c_n$ arises from a compact endomorphism on E'_n just as c_n does, and moreover, so does its positive square root g. Since this gives $(dg)(dg)^* = (b-\epsilon)_+$, we get that the partial isometry in the bidual of the C^*-algebra of compact endomorphisms on E, arising from the polar decomposition of dg determines an isomorphism between a submodule of E_n, and the submodule of E generated by $(b-\epsilon)_+$ (namely, the closure of the range of $(b-\epsilon)_+$. Since $(b-\epsilon)_+$ acts as the identity on G', G' is a submodule of the submodule generated by $(b-\epsilon)_+$, and therefore isomorphic to a submodule of (the submodule of) E'_n.

Now since any compactly contained submodule of E'_n is isomporphic to a compactly contained submodule of F, by the assumption that F is a supremum for $\{E'_n\}$, then $(G \cong)G' \cong G'' \subseteq F$, with the necessary compact endomorphism on F. Thus arbitrariness of $G \subset\subset E$ gives $[E] \leq [F]$, so $[E]$ is a supremum for $\{E_n\}$ as desired. □

Showing next that each object in $\mathcal{C}u(A)$ is the limit of a concretely rapidly increasing sequence, this last proposition will yield suprema for general increasing sequences rather quickly, so

Lemma 5.3. *For any $[E] \in \mathcal{C}u(A)$, the set of $[F]$ with $[F] \subset\subset [E]$ is upward directed (with respect to concrete compact containment) and contains a concretely rapidly increasing sequence with supremum $[E]$.*

Proof. Given a countably generated Hilbert A-module, E, take $\{\xi_n\}$ to be a generating sequence for E with $||\xi_i|| = 2^{-i}$. Noting that $\xi_i\xi_i^* : \zeta \mapsto \xi_i\langle\xi_i, \zeta\rangle$ is a positive endomorphism on E, the sum $\sum \xi_i\xi_i*$ must be a strictly positive element of the algebra of compact endomorphism on E (note that if f is a positive functional, zero on $\xi_i\xi_i^*$ for all i, then $\xi_i aa^*\xi_i^* \leq ||a||^2\xi_i\xi_i*$ gives the corresponding inner product on E to be zero on $\xi_i A$ for every i, and therefore zero on all of E. Consequently f is zero on every $\zeta\zeta^*$, and zero on the entire algebra of compact endomorphisms on E. Thus, the series $\sum \xi_i\xi_i^*$ and functional calculus allow us to construct a countable approximate unit $\{u_n\}$ for the C^*-algebra of compact endomorphisms on E such that $u_{n+1}u_n = u_n$ for every n. Then the increasing sequence

$$E_1 = \overline{u_1 E} \subseteq E_2 = \overline{u_2 E} \subseteq \cdots \subseteq E$$

has the properties that $E_n \subset\subset E_{n+1}$ for all n, and that $\bigcup E_n$ is dense in E. Consequently, $\{[E_n]\}$ is a concretely rapidly increasing sequence with supremum $[E]$, by the previous proposition. Further to this, the proof of the proposition gives that for any $[G] \subset\subset [E]$, there is an $[E_n]$ in the sequence with $[G] \subset\subset [E_n]$; taking the larger n for any two such classes of modules, provides the upward directedness of the set of such $[G]$ and completes the proof. □

Lemma 5.4. *Any increasing sequence in $\mathcal{C}u(A)$ has a supremum.*

Proof. Given a sequence $[E_1] \leq [E_2] \leq \cdots$ in $\mathcal{C}u(A)$, we have for each representative E_i a sequence:

$$E_{i1} \subset\subset E_{i2} \subset\subset \cdots E_i$$

with the additional condition that

$$[E_i] = sup_j [E_i j]$$

Noting that for any i, n, $E_{in} \subset\subset E_i \precsim E_{i+1}$, we have that E_{ij} is isomorphic to a compactly contained subobject of E_{i+1} and further, because E_{i+1} is the limit of the sequence $E_{i+1,j}$ there is some m so that E_{in} is isomorphic to a compactly contained subobject of $E_{i+1,m}$. Thus, we can choose m_n so that

$$[E_{11}] \subset\subset [E_{2,m_1}]$$

$$[E_{12}], [E_{2,m_1}] \subset\subset [E_{3.m_2}]$$

$$[E_{13}], [E_{2,m_1+1}], [E_{3,m_2}] \subset\subset [E_{4,m_3}]$$

$$[E_{14}], [E_{2,m_1+2}], [E_{3,m_2+1}], [E_{4,m_3}] \subset\subset [E_{5,m_4}]$$

i.e. that each $[E_{n+1,m_n}]$ compactly contains $[E_{n,m_{n-1}}]$ as well as one new term from each sequence $[E_i j]$ for each $i < n$. This then provides a concretely rapidly increasing sequence

$$[E_{11}] \subset\subset [E_{2,m_1}] \subset\subset [E_{3,m_2}] \subset\subset \cdots$$

which, because every sequence $\{[E_{ij}]\}_j$ contribues a new term to the terms concretely compactly contained by each $[E_{n+1,m_n}]$ (for each $n \geq i$), is eventually greater than any $[E_{ij}]$. Additionally, each term of this sequence is majorized by some $[E_i]$. These combined with the proposition that every concretely rapidly increasing sequence has a supremum, provide a supremum for $\{[E_{n+1,m_n}]\}$ which is also the supremum for $\{[E_i]\}$. Thus we have suprema for all increasing sequences in $\mathcal{C}u(A)$. □

At this point, we have enough framework on the concrete compact containment relation to prove that it is equivalent to the abstract compact containment relation $\ll$.

Lemma 5.5. *Given two equivalence classes of modules $[E], [F] \in \mathcal{C}u(A)$, $[E] \subset\subset [F]$ iff $[E] \ll [F]$. Additionally, given $[E] \in \mathcal{C}u(A)$, the set of elements $[G]$ satisfying $[G] \ll [E]$ is upward directed, and admits a rapidly increasing sequence with supremum $[E]$.*

Proof. To check that $[E] \subset\subset [F]$ gives $[E] \ll [F]$, take $[F']$ as above, and take a sequence $[F_1] \subset\subset [F_2] \subset\subset \cdots$ with a supremum $\sup[F_n] \geq [F]$ Following the construction from Proposition 5.2, we can construct a sequence $[F'_n]$ with $[F'_n] \leq [F_n]$, $[F_n] \leq [F'_{n+1}]$ and

$$F'_1 \subseteq F'_2 \subseteq \cdots$$

in which case, $\sup[F_n]$ can be expressed as the class of the inductive limit of F'_n (using the inclusion maps). We recall that any concretely compactly contained subobject of an inductive limit can be expressed as a subobject of a finite stage object F'_n for some n

Recalling that $[E] \leq [F']$ with $F' \subset\subset F$ and $[F] \leq \sup[F_n]$, take F'' isomorphic to F' with $F'' \subset\subset \lim_{\rightarrow} F'_n$. Then F'' is necessarily a subobject of some F'_n. This gives $[E] \leq [F''] \leq [F'_n]$ so that $[E] \leq [F'_n] \leq [F_n]$. This is the eventual majorization of $[E]$ by the sequence with a supremum of at least $[F]$, as required for $[E] \ll [F]$.

To prove the reverse direction, taking that $[E] \ll [F]$, and recall that F can be expressed as the limit of a sequence

$$F_1 \subset\subset F_2 \subset\subset \cdots \subset\subset F$$

Noting that $\{[F_n]\}$ is an increasing sequence with supremum at least $[F]$, then there is a

k such that $[E] \leq [F_k]$. Since $F_k \subset\subset F$ from the construction of the sequence, it follows from the definition of concrete compact conatinment that $[E] \subset\subset [F]$.

Consequently, by Lemma 5.3, we also have that any element in $\mathcal{C}u(A)$ can be expressed as the supremum of a rapidly increasing sequence, and that the set of elements it compactly contains, is upward directed. □

Now in order to have $\mathcal{C}u(A)$ as an ordered semigroup, rather than merely an ordered set, it is still necessary that addition (i.e. direct summation) be compatible with the preorder relation on the Hilbert modules.

Lemma 5.6. *$\mathcal{C}u(A)$ is an ordered semigroup, with the addition operation arising from direct summation of Hilbert A-modules. In other words, if E_1, E_2, F_1, and F_2 are Hilbert A-modules satisfying $E_1 \precsim F_1$ and $E_2 \precsim F_2$, then $E_1 \oplus E_2 \precsim F_1 \oplus F_2$.*

Proof. Take E to be a compactly contained subobject of $E_1 \oplus E_2$, i.e. $E \subset\subset E_1 \oplus E_2$. In fact, upward directedness of compactly containe dsubobjects of $E_1 \oplus E_2$ gives that there is an E' with $E \subset\subset E' \subset\subset E_1 \oplus E_2$. Now taking E_1 and E_2 to be the limits of rapidly increasing sequences of compactly contained subobjects:

$$E_1^1 \subset\subset E_1^2 \subset\subset \cdots \subset\subset E_1$$
$$E_2^1 \subset\subset E_2^2 \subset\subset \cdots \subset\subset E_2$$

and note that $E_1 \oplus E_2$ is the limit of the rapidly increasing sequence of submodules

$$E_1^1 \oplus E_2^1 \subset\subset E_1^2 \oplus E_2^2 \subset\subset \cdots \subset\subset E_1 \oplus E_2$$

Noting that Lemma 5.5 gives $[E'] \ll [E_1 \oplus E_2]$ (because $E \subset\subset E_1 \oplus E_2$), and that $\{[E_1^n \oplus E_2^n]\}$ is an increasing sequence with a supremum of at least $[E_1 \oplus E_2]$, so there is a k satisfying that $[E'] \leq [E_1^k \oplus E_2^k]$. Thus we have, by compact containment of E in E', that E is isomorphic to a compactly contained subobject of $E_1^k \oplus E_2^k$.

Now because we have $E_1 \precsim F_1$ with $E_1^k \subset\subset E_1$, then there is a $F_1' \subset\subset F_1$, which is isomorphic to E_1^k. Similarly, we have $E_2^k \cong F_2' \subset\subset F_2$. Composing the direct sum of the isomorphisms between E_i^k and F_i' with the isomorphism between E and a compactly contained subobject of $E_1^k \oplus E_2^k$, the image of E under this composition is a compactly contained subobject of $F_1' \oplus F_2'$; compact containment being transitive, it is also compactly contained in $F_1 \oplus F_2$, giving $E_1 \oplus E_2 \precsim F_1 \oplus F_2$ as desired, and enabling $\mathcal{C}u(A)$ to be well defined as an ordered semigroup. □

Now that $\mathcal{C}u(A)$ is an ordered semigroup with an additional (order theoretic) compact containment relation, it stands to verify that it is an object in $\mathcal{C}$.

Theorem 5.7 (Coward, Elliott, Ivanescu). *The equivalence classes of countably generated Hilbert A-modules forms an object in $\mathcal{C}$ for any C^*-algebra A (which we'll denote by $\mathcal{C}u(A)$). In particular*

1. *given equivalence classes of modules $[E_1]$, $[E_2]$, $[F_1]$ and $[Y_2]$ satisfying $[E_1] \leq [F_1]$ and $[E_2] \leq [F_2]$, then $[E_1 \oplus E_2] \leq [F_1 \oplus F_2]$*

2. *$\mathcal{C}u(A)$ has a zero element*

3. *every increasing sequence in $\mathcal{C}u(A)$ has a supremum in $\mathcal{C}u(A)$*

4. *every element in $\mathcal{C}u(A)$ can be expressed as the supremum of a rapidly increasing sequence $[E_1] \ll [E_2] \ll \cdots$*

5. *the operation of passing to suprema is compatible with addition, i.e.* $\sup([E_i] + [F_i]) = \sup([E_i]) + \sup([F_i])$

6. *the relation $\ll$ is compatible with addition, i.e. $[E_1] \ll [F_1]$, $[E_2] \ll [F_2]$ gives $[E_1] + [E_2] \ll [F_1] + [F_2]$.*

Proof. To get the first property, just take the class of the module $\{0\}$. The second and third properties follow from Lemma 5.4 and Lemma 5.5 respectively. Thus it only remains to show that the relation $\ll$, and the operation of passing to suprema, are both compatible with addition.

Given two increasing sequences $\{[E_n]\}$ and $\{[F_n]\}$, with suprema E and F respectively, recall that both $[E]$ and $[F]$ can be represented by the inductive limits of increasing sequences $\{E'_n\}$ and $\{F'_n\}$ of modules, and that these sequences intertwine with $\{[E_n]\}$ and $\{[F_n]\}$, giving identical suprema (also, their direct sums intertwine by compatibility of order with addition). Further, the inductive limit of the direct sums $E'_n \oplus F'_n$ is necessarily the direct sum of the limits of the individual sequences. Noting that the suprema of the classes of the modules in this case are the classes of the limits we get

$$\sup[E_n] \oplus \sup[F_n] = \sup[E'_n] \oplus \sup[F'_n] = \sup([E'_n] \oplus [F'_n]) = \sup([E_n] \oplus [F_n])$$

which is the desired compatiblity condition.

For the compatiblity of compact containment with addition, we recall that compact containment is equivalent to concrete compact containment, and check that $[E_1] \subset\subset [F_1]$ and $[E_2] \subset\subset [F_2]$ give $[E_1 \oplus E_2] \subset\subset [F_1 \oplus F_2]$. Taking F'_1 and F'_2 with $[E_i] \leq [F'_i]$ and $F'_i \subset\subset F_i$, it follows immediately that $[E_1] \oplus [E_2] \leq [F'_1] \oplus [F'_2]$ and $F'_1 \oplus F'_2 \subset\subset F_1 \oplus F_2$, so the compatibility condition is satisfied. □

Noting that the conditions in the last proof are not exactly the conditions given in Chapter 4, but are equivalent to them (simply ignore the stubs of the sequences that are not yet (rapidly) increasing), we have that $\mathcal{C}u(A)$ considered as an ordered semigroup, is an object in $\mathcal{C}$ and will henceforth use the notation $\mathcal{C}u(A)$ to refer to this object (rather than the ordered set).

5.2 Functoriality

Having that the construction $\mathcal{C}u(A)$ can be seen as a map taking C^*-algebras to objects in $\mathcal{C}$, it stands to reason that C^*-algebra homomorphisms may be mapped to $\mathcal{C}$-morphisms, in such a way as to provide that $\mathcal{C}u$ is a functor from the category of C^*-algebras to $\mathcal{C}$.

First, we need a morphism map, so given a C^*-algebra homomorphism $\phi : A \to B$, and a Hilbert A-module E_A, set $[\mathcal{C}u(\phi)](E_A)$ to be the Hilbert B-module defined by completing $(E_A) \otimes_A ({}_AB)$ with respect to the inner product

$$\langle \sum \xi_i \otimes b_i, \sum \xi'_j \otimes b'_j \rangle_B = \sum_{i,j} b_i^* \langle \xi_i, \xi'_j \rangle_A b'_j,$$

where the homomorphism ϕ enables us both to consider B as a left A-module (where the action of $a \in A$ on B is left multiplication by $\phi(a)$). Noting that the tensor product is a universal bilinear map, considering E to be $\overline{aA}$, and $a' \in E$, the universal property on the map $a' \otimes b \mapsto \phi(a')b \in B$ gives that the tensor product $(E_A) \otimes_A ({}_AB)$ is isomorphic to the B-module $\phi(a)B$, so the functorial images of homomorphisms are exactly what would be expected to agree with a positive element formulation of the semigroup.

Noting that when E_A is countably generated, $[\mathcal{C}u(\phi)](E_A)$ is also countably generated, it remains to show that the candidate morphism $\mathcal{C}u(\phi)$ is a $\mathcal{C}$-morphism; i.e. that it preserves the additive and order structure.

For this, note that given a Hilbert A-module morphism $E_A \to F_A$, the construction above admits a Hilbert B-module morphism $(E_A) \otimes_A ({}_AB) \to (F_A) \otimes_A ({}_AB)$ (since the A-module morphism preserves inner product, the definition of inner product provided above will preserve the inner product in the B-module morphism; consequently, the image morphism is isometric on the tensor product, and by continuity, isometric on their completion, i.e. the the entire modules). In particular, this provides preservation of isomorphisms, and of inclusions.

The inclusions of interest are naturally, compact inclusions. Taking E_A to be compactly contained in F_A, we'll need that the inclusion of E_B in F_B to admit a compact endomorphism on F_B which acts as the identity upon $E_B \subseteq F_B$. From the compact inclusion of E_A in F_A, we can take an endomorphism t on F_A which acts as the identity on E_A. In order to get that its push-forward image t_B acts as the identity on E_B, begin by requiring that $t_B(\eta \otimes b) = \big(t(\eta)\big) \otimes b$, for $\eta \in F_A$ and $b \in B$. Because $\langle t\eta, t\eta\rangle_A \leq ||t^*t||\langle \eta, \eta\rangle_A$, t_B must be bounded. Replacing η with $\eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_k \in F_A \otimes \cdots \otimes F_A$, a similar construction provides that the resulting map is still bounded. This then allows us to observe that if t is a sum of finite-rank endomorphisms $\zeta\zeta^*$, then so is t_B; and likewise for the limits of such finite rank endomorphisms. Consequently, when t is a compact endomorphism, then so is t_B, and so compact containment is preserved by this module map.

With isomorphism and inclusion already preserved by the module map $\mathcal{C}u(\phi)$, the addition of compact containment provides also preservation of the preorder used to construct $\mathcal{C}u(A)$ (said preorder simply being a construction of inclusions, isomorphisms, and compact containment). The relation $\ll$ being defined in terms of the preorder, it follows that it is also preserved, thus $\mathcal{C}u(\phi)$ is a morphism in the category $\mathcal{C}$. $\mathcal{C}u(\phi)$ being a natural morphism (note that the identity isomorphism from a C^*-algebra take each module to the class of scalar multiples – to the extent that operators can be considered scalar – of itself, i.e. itself), we get functoriality of $\mathcal{C}u$.

5.3 Continuity under inductive limits

Since this functor $\mathcal{C}u$ is intended to serve as an equivalent to the established Cuntz semigroup, but preserve inductive limits, it only makes sense to check that it does in fact preserve inductive limits. To show this, given a sequence of C^*-algebras $A_1 \to A_2 \to \cdots$ with inductive limit A, we need to show that every Hilbert A-module E is equivalent to the supremum of the increasing part of an eventually increasing sequence of images of $e_i \in \mathcal{C}u(A_i)$ in $\mathcal{C}u(A)$.

Towards this end, note that Theorem 2.5 gives that if E is countably generated, it must be isomorphic to a submodule of the direct sum $\bigoplus^\infty A$ of countably infinitely many copies of A, which we can consider as containing E itself (since isomorphic modules are necessarily equivalent). Considering that A might not be countably generated, take A' to be a countably generated closed submodule of A so that $E \subseteq \bigoplus^\infty A'$ (which can be done, since E is countably generated). Using G to denote $\bigoplus^\infty A'$, we get the following:

Proposition 5.8. *There exists G containing E as above, and there is a sequence $G_1 \subseteq G_2 \subseteq \cdots \subseteq G$ of subobjects of G such that each G_n arises from the functorial push-forward of some module in A_n to A.*

Proof. Take such an G and note that since $A' \cap A_n$ is necessarily contained in the push-forward of A_n (considered as a module over itself), we can consider the modules to be right modules giving a sequence $(A' \cap A_1)A, (A' \cap A_2)A, \ldots$ as generating A' as the closure of its union (if it doesn't, expand A' to adjoin countably many elements of each A_i which approximate the generators of A'; retaining that A' is countably generated, and contains E). G being the countable direct sum of copies of A', G_n can then be taken as the sum $G_n = \bigoplus^\infty (A' \cap A_n)A$. □

Since the G_n in this sequence arise as modules on A_n, let's use G_n to denote the finite state modules, and $(G_n)_A$ to denote their images under the push-forward map to $\mathcal{C}u(A)$. Now, recalling that each compact endomorphism on G arises as a countable series of elementary endomorphisms $\xi\zeta^*$, and since every $\xi, \zeta \in G$ are approximated by some element of $(G_n)_A$, every compact endomorphism on G arises as the limit of some sequence of compact endomorphisms on G_n. This allows us to move to the following:

Lemma 5.9. *Given a Hilbert A-module F, there is an increasing sequence $f_1 \leq f_2 \leq \cdots$*

in $\mathcal{C}u(A)$*, with each* f_i *arising from an element of* $\mathcal{C}u(A_i)$*, with each* $f_i \leq f_{i+1}$ *in* A_{i+1} *and with* $\sup f_i = [F]$*.*

Proof. Continuing with F taken to be a subobject of G, take h to be a strictly positive compact endomorphism on F and recall that it extends naturally to a compact endomorphism on G. Using the sequence from the previous proposition, and taking the push-forward images in A, we getthat $G = \overline{\bigcup (G-i)_A}$ and that every compact endomorphism on G can be approximated (in norm) by some push-forward of a compact endomorphism on some G_n. Thus we can take, for any compact endomorphism h on G, a series of compact endomorphisms $h_n \in G_n$ (we'll also use h_n to denote their push-forward images $(h_n)_A$ in $\mathcal{K}(G)$), with $h_n \to h$ (in $\mathcal{K}(G)$). Then taking $\epsilon > 0$, and n so that $||h - h_n|| < \epsilon$, Lemma 2.4 gives

$$(h_n - \epsilon)_+ = dhd^*$$

for some some compact endomorphism d over G. Moreover, recalling from the proof of Lemma 2.4, the construction of d, we get that $h^{\frac{1}{2}}d^*dh^{\frac{1}{2}} = yy^*$ where $y = v(h_n - \epsilon)_+^{\frac{1}{2}}$ giving $yy^* = vx^*xv* = (v|x|)(v|x|)^* = xx^*$ from the polar decomposition of x. Additionally, x was defined to be equal to $h_r^{\frac{1}{2}}e_r$ (the index r has been added here to represent the $r > 1$ chosen to define g_r) which, depending on the choice of r, can give h_r arbitrarily close to h, and e_r arbitrarily close to a compact operator acting as the identity on h (both as r is chosen close to 1). Thus $h^{\frac{1}{2}}d^*dh^{\frac{1}{2}} = xx^* = h_r^{\frac{1}{2}}e^2h_r^{\frac{1}{2}}$ can be made arbitrarily close to h. Let us say instead then, that h is within ϵ of $h^{\frac{1}{2}}d^*dh^{\frac{1}{2}}$; also a compact endomorphism on F. Once again using Lemma 2.4, we get a compact endomorphism e on F satisfying

$$(h - \epsilon)_+ = eh^{\frac{1}{2}}d^*dh^{\frac{1}{2}}e^*$$

Noting that the partially isometric part of $dh^{\frac{1}{2}}e^*$ is an isometry from $\overline{(h-\epsilon)_+F}$ to $\overline{(h_n-\epsilon)_+G}$ with image $(dh^{\frac{1}{2}}e^*eh^{\frac{1}{2}}d^*G)^- \subseteq \overline{(h_n-\epsilon)_+G}$, it follows that $\overline{(h-\epsilon)_+F}$ (a compactly contained subobject of F) is isomorphic to a compactly contained subobject of $\overline{(h_n - \epsilon)_+ G}$. This gives in $\mathcal{C}u(A)$, that

$$\left[\overline{(h-\epsilon)_+ F}\right] \leq \left[\overline{(h_n - \epsilon)_+ G}\right].$$

Recalling that G is the closure of the union of $\{(G_n)_A\}$, we get that each $\overline{(h_n - \epsilon)_+ G}$ is the closure of the union of the increasing subobjects $\overline{(h_n - \epsilon)_+ (G_k)_A}$, each one of which arises as the push-forward of the finite stage object $\overline{(h_n - \epsilon)_+ (G_k)_{A_l}}$ where $l = \max(k, n)$. Recalling that the supremum of a sequence of classes of modules with an increasing sequence of representatives, is the class of the closure of the union of those representatives, we get that

$$\left[\overline{(h_n - \epsilon)_+ G}\right] = \sup_k \left[\overline{(h_n - \epsilon)_+ (G_k)_A}\right].$$

Putting this aside for a moment, and recalling that h is a strictly positive compact endomorphism on F, we get that $hF = F$ and, taking ϵ_m to be a sequence tending strictly monotonically to 0 we additionally get

$$F = \overline{\bigcup \overline{(h-\epsilon)_+ F}}$$

which by a similar argument to the one above gives, in $\mathcal{C}u(A)$:

$$[F] = \sup_m \left[\overline{(h-\epsilon)_+ F}\right]. \tag{5.1}$$

Returning to the examples increasing to G, and noting that the definition of h as strictly positive on F gives $F = \overline{hG}$, the construction of $(h_n - \epsilon)_+ = dhd^*$ provides an isomorphism (namely the partially isometric part of the compact homomorphism $h^{\frac{1}{2}} d^*$) between $\overline{(h_n - \epsilon)_+ G}$ and a subobject of F. Taking the compact homomorphism arising from a smaller ϵ, we get that the image of $\overline{(h_n - \epsilon)_+ G}$ in F is a compactly contained subobject, so

$$\left[\overline{(h_n - \epsilon)_+ G}\right] \ll [F] \tag{5.2}$$

in $\mathcal{C}u(A)$.

Now consider the secuence of classes of modules on the subsequence $A_{k_1} \to A_{k_2} \to \cdots$ of $A_1 \to A_2 \to \cdots$ described by

$$g'_m = \left[\overline{(h_{n_m} - \epsilon_{l_m})_+ (G_{k_m})_A}\right], \quad m = 1, 2, \ldots$$

where the sequences $\{n_m\}, \{l_m\}$, and $\{k_m\}$ are constructed as follows.

First, recalling the sequence $\{\epsilon_m\}$, strictly monotonically decreasing to 0, use 5.2 to choose n_1 so that

$$\left[\overline{(h - \epsilon_2)_+ F}\right] \leq \left[\overline{(h_{n_1} - \epsilon_2)_+ G}\right].$$

Since $\epsilon_2 < \epsilon_1$, it follows that $\overline{(h_{n_1} - \epsilon_2)_+ G} \subset\subset \overline{(h_{n_1} - \epsilon_2)_+ G}$, and consequently we have compact containment of the equivalence classes (in the order-theoretic sense by its equivalence with the concrete sense). This compact containment then allows k_1 to be chosen so that

$$\left[\overline{(h_{n_1} - \epsilon_2)_+ G}\right] \leq \left[\overline{(h_{n_1} - \epsilon_2)_+ (G_{k_1})_A}\right].$$

The latter class being compactly contained in $[F]$ ($(G_{k_1})_A$ being a subobject of G), l_2 can then be chosen (considering that $l_1 = 1$) so that

$$\left[\overline{(h_{n_1} - \epsilon_1)_+ (G_{k_1})_A}\right] \leq \left[\overline{(h - \epsilon_{l_2+1})_+ F}\right].$$

Now n_2 can be chosen as above so that

$$\left[\overline{(h - \epsilon_{l_2+1})_+ F}\right] \leq \left[\overline{(h_{n_2} - \epsilon_{l_2+1})_+ G}\right].$$

and since $\epsilon_{l_2+1} < \epsilon_{l_2}$, compactness allows k_2 to be chosen in a similar manner to k_1 to get

$$\left[\overline{(h_{n_2} - \epsilon_{l_2+1})_+ G}\right] \leq \left[\overline{(h_{n_2} - \epsilon_{l_2})_+ (G_{k_2})_A}\right].$$

Now with l_2 already having been chosen as a successor to 1, we see that this process can be continued to obtain an increasing sequence $g'_m = \left[\overline{(h_{n_m} - \epsilon_{l_m})_+ (G_{k_m})_A}\right]$ in $\mathcal{C}u(A)$, which is intertwined with the increasing sequence $f'_m = \left[\overline{(h-\epsilon_{l_m+1})_+F}\right]$. Noting that 5.1 gives $[F]$ as the supremum for $\{f'_m\}$, we get from the intertwining that $[F]$ is also the supremum for $\{g'_m\}$. Noting that g'_m is the pushforward image of $\left[\overline{(h_{n_m}-\epsilon_{l_m})_+G_{k_m}}\right]$ from G_{k_m}, and taking $f_n = g'_m$ whenever $k_m \leq n < k_{m+1}$, we get that each f_n arises from an element of $\mathcal{C}u(A_n)$, and that $\{f_n\}$ is increasing in (A), with supremum $[F]$. Thus, it remains only to show that for every n, $f_n \leq f_{n+1}$ in $\mathcal{C}u(A_{n+1})$ (equivalently, that $g'_m \leq g'_{m+1}$ in $\mathcal{C}u(A_{k_{m+1}})$).

Remembering $\{h_m\}$ to be increasing to h, and $\{\epsilon_m\}$ to be strictly decreasing to 0, it follows that $h_{n_m} - (h_{n_{m+1}} - \epsilon_{l_{m+1}})_+ < \epsilon_{l_m}$, and that the left side being positive from the same monotonicity, the conditions of Lemma 2.4 are satisfied to give a compact endomorphism d on $G_{n_{m+1}} \subseteq G_{k_{m+1}}$ (which doesn't require refinement, as the selection of k_m so that h_{n_m} can act upon G_{k_m} already requires $k_m \geq n_m$) and this endomorphism satisfies

$$(h_{n_m} - \epsilon_{l_m})_+ = d(h_{n_{m+1}} - \epsilon_{l_{m+1}})_+ d^*.$$

Taking a partial isometric part as before, it follows that $\overline{(h_{n_m} - \epsilon_{l_m})_+ G_{k_m}}$ is isomorphic to a subobject of $\overline{(h_{n_{m+1}} - \epsilon_{l_{m+1}})_+ G_{k_{m+1}}}$, over $A_{k_{m+1}}$, which gives $g'_m \leq g'_{m+1}$ in $\mathcal{C}u(A_{k_{m+1}})$ as required. Consequently the sequence $\{f_n\}$ has the desired properties. □

Having now demonstrated $\mathcal{C}u(A)$ to consist entirely of the equivalence classes of suprema of increasing sequences on $\{\mathcal{C}u(A_n)\}$, we need only show that the order relation defined on $lim_{\rightarrow}\mathcal{C}u(A_i)$ is identical to the order relation on $\mathcal{C}u(A)$ in order to prove the main result of this thesis:

Theorem 5.10 (Coward, Elliott, Ivanescu). *The functor $\mathcal{C}u$ from the category of C^*-algebras to $\mathcal{C}$ preserves inductive limits (i.e. if $A_1 \to A_2 \to \cdots$ has inductive limit A, then $\mathcal{C}u(A_1) \to \mathcal{C}u(A_2) \to \cdots$ has inductive limit $\mathcal{C}u(A)$).*

Proof. To get equivalence of the order relations, it sufficient to show that given $e_1 \leq e_2 \leq$

$\cdots$ and $f_1 \leq f_2 \leq \cdots$ with $e_i, f_i \in \mathcal{C}u(A_i)$, then $\sup e_i \leq \sup f_i$ in $\mathcal{C}u(A)$ iff, whenever $g \ll e_i$ in $\mathcal{C}u(A_i)$, there is a $j \geq i$ with $g \ll f_j$ in $\mathcal{C}u(A_j)$.

Because $\mathcal{C}u(A)$ is already established to be an object in $\mathcal{C}$, we know that each element can be expressed as the supremum of a rapidly increasing sequence and, through a diagonal argument similar to those presented before, take the elements of the rapidly increasing sequence to be the images of elements from the building block algebras. Consqeuently, we can assume that the sequences $\{e_i\}$ and $\{f_i\}$ are rapidly increasing (by replacement with a sequence having the same supremum). Moreover, we can take, for each e_i and f_i, that $e_i = [E_i]$ and $f_i = [F_i]$, where E_i and F_i are Hilbert A_i-modules, and also

$$E_1 \subset\subset E_2 \subset\subset \cdots, \qquad F_1 \subset\subset F_2 \subset\subset \cdots$$

so we have

$$\sup e_i = \big[lim_{\rightarrow}(E_i)_A\big] \text{ and } \sup f_i = \big[lim_{\rightarrow}(F_i)_A\big].$$

Assuming that $\sup e_i \leq \sup f_i$ in $\mathcal{C}u(A)$, we need to show that for any i, and $g \ll e_i$ in $\mathcal{C}u(A_i)$, there is a j with $g \ll f_j$ in $\mathcal{C}u(A_j)$. To this end, let us also take such an i and g. Then Lemma 5.5 provides that the concrete compact containment relation $g \subset\subset e_i$ also holds; i.e. there is some $G^1 \subset\subset E_i$ over A_i, with $g \leq [G_1]$. Additionally, we can use upward-directedness of compactly contained elements, and the existence of rapidly increasing sequences in $\mathcal{C}u(A_i)$ to obtain G^2, G^3 satisfying

$$G^1 \subset\subset G^2 \subset\subset G^3 \subset\subset E_i \text{ over } A_i.$$

Pushing this forward to the space of Hilbert A-modules, we get

$$G^1_A \subset\subset G^2_A \subset\subset G^3_A \subset\subset (E_i)_A$$

Noting that Lemma 5.5 provides $[G^3] \leq [E_i] \ll \sup e_i \leq \sup f_i$, the definition of (order-theoretic) compact containment provides a j with $[G^3] \leq [F_j]$ in $\mathcal{C}u(A)$.

G^2 being a compactly contained subobject of G^3, the definition of the order relation on $\mathcal{C}u(A)$ then provides that it must be isomorphic to a compactly contained subobject $(F_j')_A$ of $(F_j)_A$. Additionally, with $G^1 \subset\subset G^2$, we know there to be a compact endomorphism h on G^2 which is the identity on $G^1 \subset G^2$. Moreover, it can be chosen to be positive and, chosen with ϵ, so that $(h-\epsilon)_+$ acts as the identity on G^1.

Since h is a positive element, and $G_A^2 \cong (F_j')_A \subseteq (F_j)_A$, we can find a compact homorphism x from G_A^2 to $(F_j)_A$ so that $h = x^*x$ (particularly, the product of $h^{\frac{1}{2}}$ with the isomorphism between the subobjects). Taking $k \geq \max(i,j)$, we get that both F_j and G^2 can be pushed forward to A_k-modules. A homomorphism x' can then be taken between these A_k-modules whose push-forward approximates x in norm, in the algebra of compact homomorphisms from G_A^2 to $(F_j)_A$. Consequently, we have that x'^*x' is close to h in A, and that both such endomorphisms arise from compact endomorphisms on $G_{A_k}^2$ (possibly earlier, but we can push them forward). This then allows us to take that they're (almost as) close over A_l for some $l \geq k$, say within the ϵ chosen above (so that $(h-\epsilon)_+$ actas as the identity on G^1). Then because $||h - x'^*x'|| < \epsilon$, Lemma 2.4 provides an endomorphism d so that

$$(h-\epsilon)_+ = d^*x'^*x'd \quad (\text{in } \mathcal{K}(G_{A_l}^2).$$

Replacing x' with $x'd$, we get x' to be a homomorphism between $G_{A_l}^2$ and $(F_j)_{A_l}$, where $x'^*x' = (h-\epsilon)_+$, which acts as the identity on $G_{A_l}^1$.

Now restricting x' to $G_{A_l}^1$, its image in $(F_j)_{A_l}$ is necessarily isomorphic to it (from x'^*x' acting as the identity on $G_{A_l}^1$), and further, $x'x'^*$ is a compact endomorphism on $(F_j)_{A_l}$ which acts as the identity on this image. Thus, $[G^1] \ll [F_j]$ which, since $[G^1]$ was arbitrarily chosen to be compactly contained in $[E_i]$, provides $[E_i] \leq [F_j]$ as needed.

This leaves the reverse implication to be shown: suppose that whenever g is chosen to that $g \ll e_i$ for some i, we get a $j \geq i$ admitting $g \ll f_j$; we need that $\sup e_i \leq \sup f_i$

in $\mathcal{C}u(A)$. To prove that $\sup e_i \leq \sup f_i$, we need only show that for any e_i, there is a $f_j \geq e_i$; to this end, take e_i, and observe that since $\mathcal{C}u(A_i)$ is an object in $\mathcal{C}$, we can construct an increasing sequence $\{g_n\}$ with $g_n \ll e_i$ for every n, and $e_i = \sup g_n$. Our hypothesis then gives that $g_n \ll f_j$ for every n in $\mathcal{C}u(A_i)$, and by functoriality in $\mathcal{C}u(A)$. Consequently f_j is an upper bound for $\{g_n\}$ in $\mathcal{C}u(A)$, and since functoriality also gives $e_i = \sup g_n$ in $\mathcal{C}u(A)$, it follows that $e_i \leq f_j$ in $\mathcal{C}u(A)$, so we are done.

□

chapterComparing the semigroups

Now we have, for any C^*-algebra A, a semigroup $\mathcal{C}u(A)$, a category $\mathcal{C}$ to which this semigroup belongs, and some nice properties for this category, and for the functor $\mathcal{C}u$, in addition to the established Cuntz semigroup $W(A)$. Naturally, it makes sense to compare the two semigroups and see whether or not $\mathcal{C}u$ can reasonably be said to be a Cuntz semigroup functor.

A cursory examination shows that if we hold ourselves strictly to the definition of $W(A)$, then $\mathcal{C}u(A)$ and $W(A)$ differ on the C^*-algebra $\mathbb{C}$! To see this simply note that $W(\mathbb{C}) = \mathbb{N}$, whereas $\mathcal{C}u(\mathbb{C})$ is exactly the collection of isomorphism classes of complex Hilbert spaces; one for each dimension in $\mathbb{N}$, and also the countable dimensional space $\mathcal{H}_{\mathbb{C}}$; so we have $\mathcal{C}u(\mathbb{C}) = \mathbb{N} \cup \{\infty\}$. We can recall however, that this distinction was exactly the distinction between $W(A)$ defined as equivalence classes of positive elements in $M_\infty(A)$, and a hypothetical semigroup defined as equivalence classes of positive elements in $A \otimes \mathcal{K}$. Moreover, this latter semigroup is isomorphic to $W(A \otimes \mathcal{K})$ (which is isomorphic to $W(A)$ when A is stable). Consequently, we can consider $\mathcal{C}u(A)$ to be a stable Cuntz semigroup, if we can demonstrate the semigroup isomorphisms

$$\mathcal{C}u(A) \cong \mathcal{C}u(A \otimes \mathcal{K}) \cong W(A \otimes \mathcal{K}).$$

Proposition 5.11. *For any C^*-algebra A, $\mathcal{C}u(A)$ is isomorphic to $\mathcal{C}u(A \otimes \mathcal{K})$, and further, this isomorphism arises from a natural transformation.*

Proof. Begin by considering the natural inclusion map $A \to A \otimes e \subseteq A \otimes \mathcal{K}$ (for some fixed, rank one projection e in $\mathcal{K}$). Then because $\mathcal{C}u$ is a functor, this induces a (natural) morphism $\mathcal{C}u(A) \to \mathcal{C}u(A \otimes \mathcal{K})$. Thus it is only necessary to demonstrate that this $\mathcal{C}$ morphism is an isomorphism.

To do this, consider a countable collection of rank one projections $\{e_i\}$ in $\mathcal{K}$, with $e_1 = e$, and whose direct sum acts as the identity upon $\mathcal{K}$. Being rank one, all of these projections are Murray-von Neumann equivalent, so the Hilbert $A \otimes \mathcal{K}$-module images of A under the $\mathcal{C}$ homomorphisms induced by the C^*-algebra morphisms $A \mapsto A \otimes e_i$, are all isomorphic, and therefore Cuntz equivalent. Further, compatibility between direct summation and Cuntz equivalence then gives that, for a family of A-modules $\{F^i\}$, we can take F^i_j to be the image of the A-module F^i under the map induced by $A \mapsto A \otimes e_j$ (roughly F^i in the e_j coordinate of an infinite row vector, which is zero elsewhere), and get

$$\bigoplus_i F^i_i = \bigoplus_i F^i_1$$

The cut-down of this latter module by the algebra $A \otimes e_1$ then provides an inverse $\mathcal{C}$ morphism for the $A \mapsto A \otimes e_1$ morphism and, since the latter is a morphism by functoriality of $\mathcal{C}u$, and because the inverse works both ways, we have our isomorphism between $\mathcal{C}u(A)$ and $\mathcal{C}u(A \otimes \mathcal{K})$. □

The first isomorphism having been proven, let us now move on to the second. For the sake of clarity, we'll observe that $A \otimes \mathcal{K}$ is always stable, and cut down on the complexity of the notation by assuming that A must be stable.

Theorem 5.12 (Coward, Elliott, Ivanescu). *Given a stable C^*-algebra A, let ϕ be the map taking each positive element $a \in A$ to the right ideal $\overline{aA}$, considered as a Hilbert A-module. Then ϕ induces a bijection between $W(A)$ and $\mathcal{C}u(A)$ which preserves the order relation in both directions. Further, the map from $W(A)$ to $\mathcal{C}u(A)$ induced by ϕ is an (order) isomorphism between $W(A)$ and $\mathcal{C}u(A)$.*

Proof. We'll begin by showing the preservation of preordering in the forward direction: take a, b to be positive elements in $M_n(A)$ (but since A is stable, we can use the isomorphism between $M_n(A)$ and A to find equivalent positive elements in A, and will just refer to A from now on) with $a \precsim b$. Recall that this means we have a sequence $\{x_n\}$ in A

so that $x_n b x_n^*$ converges to a. Because of this convergence, we can take, for each n, an $\epsilon_n > 0$ with $||x_n b x_n^* - a|| < \epsilon_n$ and so that ϵ_n decreases to 0.

Now, by Lemma 2.4 we have, again for each n, d_n so that $d_n x_n b x_n^* d_n^* = (a - \epsilon_n)_+$. Recalling that $(a-\epsilon_n)_+$, which we'll call a_n, is a continuous function on a (in the spectral calculus), we have that $\{a_n\}$ is an increasing sequence of continuous functions on a, with limit a. This then gives that $\overline{aA}$ is the inductive limit, and therefore the supremum of the increasing sequence $\overline{a_1 A} \subseteq \overline{a_2 A} \subseteq \cdots$. Moreover the equivalence of concrete and order-theoretic compact containment provide that any compactly contained subobject of $\overline{aA}$ must be a compactly contained subobject of $\overline{a_n A}$ for some n. Additionally we have that, because $a_n = d_n x_n b x_n^* d_n^*$, $\overline{a_n A}$ must be isomorphic to a subobject of $\overline{bA}$; by this same isomorphism, any compactly contained subobject of $\overline{a_n A}$ is compactly contained in $\overline{bA}$, which with the other compact containment we just demonstrated, gives that $\overline{aA} \precsim \overline{bA}$ as desired.

Now we want preorder preservation in the reverse direction, so take a, b to be positive in A so that $\overline{aA} \precsim \overline{bA}$ as Hilbert A-modules. To get this, take f to be a continuous, positive, real-valued function on the spectrum of a, equal to zero in a neighbourhood of zero provided that zero is in the spectrum of a), so that $||f(a) - a||$ is small. Taking g to be a continuous function on the spectrum of a, equal to 1 where f is nonzero, and equal to 0 at zero. We have then that $\overline{f(a)A}$ is a submodule of $\overline{aA}$ and moreover that left multiplication by $g(a)$ is an endomorphism on $\overline{aA}$ which acts as the identity on $\overline{f(a)A}$. Thus $\overline{f(a)A}$ is compactly contained in $\overline{aA}$ and because $\overline{aA} \precsim \overline{bA}$, is isomorphic to a (compactly contained) submodule of $\overline{bA}$.

From this isomorphism, we can get take the square root of $f(a)$, and multiply it by a partial isometry in A^{**} representing the isomorphism, to get x so that $x^*x = f(a)$, and xx^* generates the isomorphic submodule of $\overline{bA}$. Moreover, with $\overline{xx^*A}$ as a submodule of $\overline{bA}$, we also get that there is a sequence $\{d_n\}$ so that bd_n converges to xx^*, so x^*bd_n

converges to x^*xx^*, which as we'll see in a moment, generates the same closed right ideal as $x^*x = f(a)$. In particular, we can modify d_n to drop the trailing x^* (multiplying it on the right by an approximate inverse; recall that x is the product of a positive element and a partial isometry, so we can take the adjoint of the partial isometry, and apply the function $g_n(x) = \max(x^{-1}, n)$ to the positive part to get a convergent sequence x^*bd_n with our new d_n) and get $f(a)$ as the limit of x^*bd_n. Recalling that this the follows the form of Cuntz's original defintion for the preorder relation on the Cuntz semigroup, we get that $f(a) \precsim b$. Since f can be chosen to get $f(a)$ arbitrarily close to a, we can construct a sequence $\{f_n\}$ of such f, satisfying $||f_n(a) - a|| < \frac{1}{n}$, and for each f_n, we have a sequence $\{x_i^n\}$ with $||x_i^{n*}bx_i^n - f_n(a)|| < \frac{1}{i}$, we can take a diagonal subsequence $\{x_n^n\}$ to get $a \precsim b$ as desired.

Preservation of the preorder in both directions having been shown, injectivity of the map follows. Additionally, preservation of the addition operation holds trivially, so it is now necessary only to verify that the map is surjective; i.e. that each countably generated Hilbert A-module is equivalent (in this case, equivalence will follow from isomorphism) to one that is generated, as a right ideal, by some positive element in A.

Taking a countably generated Hilbert A-module, E, Theorem 2.5 provides that it is isomorphic to a direct summand of the countably infinite direct sum $\mathcal{H}_A$. Since A is stable, the direct sum $\mathcal{H}_A$ is isomorphic to A (considered as an A-module), and so E, being a direct summand (and therefore closed submodule) of $\mathcal{H}_A$, is isomorphic to a closed submodule E' of A.

E' then being a submodule of A, closure under A multiplication provides that it is a right ideal, and further, it is a closed right ideal, being closed in its inner product topology. Additionally, because E' is a submodule of A, and A is countably generated, E' is itself countably generated as both an A-module, and a right ideal; A being stable gives then that E' is, in fact, isomorphic to a singly generated ideal (the single generator being the direct sum of the countably many generators). Taking e to be a generator for this ideal, we recall that 2.7 gives any convergent sequence in the module to be expressible with a common left factor. From this, we get that the closedness of the module can be taken as arising from taking the closure of the ideal eA, so we can write $E \cong \overline{eA}$, providing the desired equivalence, and establishing the isomorphism between $\mathcal{C}u(A)$ and $W(A)$. □

With this, we now have a framework in which the Cuntz semigroup can be taken as a functor which preserves inductive limits (at least on the subcategory of stable C^*-algebras). Additionally, we have a formulation of the Cuntz semigroup in terms of Hilbert C^*-modules, which allows us to prove some results that would be impractical in the setting of positive elements. Some of these results will follow in the next chapter.

Chapter 6

Additional results

6.1 The commutative case revisited

Now that the Cuntz semigroup has been defined in terms of modules, rather than positive elements, we can more easily check whether the trivial subsemigroup $U(C(X))$ is in fact the entire subsemigroup in certain well behaved cases. The motivation for this, is that we can consider the Hilbert $C(X)$-modules as generalized vector bundles. In particular, we have that they can be expressed as the direct sums of vector bundles over closed subspaces of X.

Since every closed subspace of a 1-dimensional CW-complex is homeomorphic to another CW-complex of dimension ≤ 1, and because every CW-complex Y of dimension ≤ 1 has $K_0(Y) = \mathbb{Z}$, there ought to be no nontrivial vector bundles on any open subspace of X. Consequently, $W(C(X))$ should be exactly the semigroup of trivial vector bundles over open subspaces of X, i.e. $U(C(X))$.

Remark If X is a 1-dimensional CW-complex, then the Cuntz semigroup $W(C(X))$ appears to consist entirely of sums (and series) of classes of positive elements in $C(X)$ (i.e. equal to $U(C(X))$).

As far as CW-complexes are concerned, this condition can probably be refined to a necessary and sufficient condition; namely that X does not contain any open subspaces Y, where Y is not homotopic to a 1-dimensional (or 0-dimensional) CW-complex. That this condition is sufficient would follow from the homotopy invariance of K_0 on all the subspaces providing an argument somewhat like above. For necessity, we would consider a open subspace Y without a suitable homotopic space as above. Then Y either is, or contains as a subspace, a closed 2-manifold M. Because M is a closed 2-manifold, its K_0 can be expected to admit a Bott projection, which is not equivalent to a trivial projection; consequently we would get a corresponding nontrivial element in the Cuntz semigroup $W(C(X))$, so

Remark Given a CW-complex X, then $W(C(X)) = U(C(X))$ is likely equivalent to the condition that X does not contain any open subspaces which are not homotopic to a 1-dimensional CW-complex.

It's also worth noting about the commutative case that the pushed forward images of nontrivial elements to inductive limits provide much of the basis for the counterexamples to K-theoretic classification of AH algebras.

6.2 Additional properties for semigroups in $\mathcal{C}$

Another point to note is that in [2], $\mathcal{C}$ is defined so that its objects have zero as a minimal element, while this is not the case for $\mathcal{C}$ as defined in this thesis. That zero is a minimal element of $\mathcal{C}u(A)$ for any C^*-algebra A is a trivial matter of taking $x_n = 0$ for every n to establish a comparison with the positive element preorder relation.

Slightly more interesting is that the compatibility of the order relation with addition on objects in $\mathcal{C}$ prevents there being any torsion subsemigroups. This in turn provides that an element less than 0 in such a semigroup will be greater than any of its (integer ≥ 2) multiples and so the only possible minimal element for such a semigroup is in fact 0.

Additionally, when the zero is a minimal element, any sequence increasing to an element greater than or equal to it necessarily majorizes it at its first entry, and so $0 \ll 0$ (though this must also hold in order for each element of the semigroup to be the supremum of a rapidly increasing sequence and so provides little value aside from an assurance that objects in $\mathcal{C}$ with zero minimal are well-defined).

We may also want to examine the conditions identified for isolating Murray-von Neumann semigroup elements in the Cuntz semigroup, both by my coauthors and me in [2], and by Toms and Perera in [14]. In the former, equivalence classes of projections (in a C^*-algebra with real rank zero) are shown to be exactly those satisfying:

$$[p] \ll [p]$$

while the latter establishes them as satisfying (in a C^*-algebra with stable rank one):

$$\forall z \gtrsim p, \exists y \text{ so that } [z] = [p] + [y].$$

In particular, since both these conditions identify projections in certain settings, are they equivalent. In the compact (as opposed to finite matrix) setting, there is the obvious counterexample of infinite elements. The only objects which majorize them are equivalent to them and so they trivially satisfy the second condition. The first condition however, fails to hold even in $\mathcal{C}u(\mathbb{C})$, since the infinite element can be expressed as the limit of any strictly increasing sequence of finite elements (none of which will majorize it, as is needed to satisfy the first condition).

Even considering the non-stable case though, we have the example of $\mathcal{C}u(C_0(0,1))$, for which the element associated with the characteristic function of $(0,1)$ itself satisfies the second condition (subtracting 1 from a lower semicontinuous functions preserves its lower semicontinuity, and provides the other summand). Further, the increasing sequence $\{\chi_{(\frac{1}{n},\frac{n-1}{n})}\}$ provides failure of the first condition.

The setting in which these counterexamples arise though, is suggestive that the compact containment condition could be seen as analogous to being a compact set in a topological space, while the summation condition would be analogous to being a closed set. In this setting, we ought to expect that compactness implies closure, but the $\mathcal{C}$ object structure doesn't necessarily provide any Hausdorff type conditions, so no proof (or possibly counterexample) is yet available.

Chapter 7

Further research

While there are plenty of opportunities to examine the structure of objects in $\mathcal{C}$, to establish details like e.g. whether a 'compact' element is necessarily 'closed' (in the sense of the end of the last chapter), or whether the sum of a 'non-compact' element with any other element is necessarily 'non-compact' (as in the case for the UHF algebra A with $K_0(A) \cong \mathbb{Q}$), there is even more to examine in terms of using the continuity of the functor $\mathcal{C}u$ under inductive limits to answer questions about C^*-algebras.

One question that naturally arises in the setting of inductive limits is how to deal with the variety of available maps between the building blocks. In particular, many interesting examples of AH algebras are constructed with point evaluations in the building block maps. The choice of point may not appear in a significant manner in the K-theory, or even the Murray-von Neumann semigroup (even at the building blocks), but the subsemigroup U identified earlier would distinguish between points selected for evaluation. In light of this, it may be worth investigating how these differences in point evaluations can be smoothed out, e.g. by a small corner argument. Such work would then also shed more light on the Cuntz semigroups of the C^*-algebras that Toms used as counterexamples for K-theoretic classification.

Additionally, given that the "trivial" subsemigroup U was shown to be the Cuntz semigroup for commutative algebras with zero and one dimensional base spaces, it would make sense to investigate whether a suitable analogue of this triviality property holds for AF, AI, and AT algebras. Further, given that the base space can be identified with the extrema on the simplex of traces, such an analogous property, having its origin in base spaces, may very well hold for algebras that are merely tracially AF, tracially AI, or tracially AT.

Another example that may warrant investigation is the Cuntz semigroup of the Jiang-Su algebra $\mathcal{Z}$ (a calculation for which is given in [1] and [14], since $\mathcal{Z}$ is $\mathcal{Z}$-stable). Given the connections between $\mathcal{Z}$-stability, dimension growth, and almost unperforation in the Cuntz semigroup, $\mathcal{C}u(\mathcal{Z})$ may bring some additional clarity to this connection. The catch here is that the functor $\mathcal{C}u$ does not seem to respect tensor products, so some sort of theory of Cuntz semigroup tensor products would need to be properly developed (not that this should seem too daunting – after all, it wasn't that long ago that the Cuntz semigroup didn't respect inductive limits!).

In a similar vein of continuing to modify the category $\mathcal{C}$, it may be possible to modify the condition requiring the existence of suprema for all increasing sequences, to merely require it for bounded increasing sequences (and correspondingly modify the functor $\mathcal{C}u$ to use finitely generated modules, instead of countably generated modules), in order that the module-based map may agree with the positive element definition on non-stable C^*-algebras.

Finally, it may be worth adapting some of the techniques for K-theoretic classifications of various types of AH algebra (in particular the techniques used in [6] and [4] may be quite valuable) to develop a Cuntz-semigroup classification theory for all AH (and possibly even ASH) algebras. Moreover, in light of the recovery of the Cuntz semigroup functorially from the Elliott invariant in [1], such a result would also provide K-theoretic classification of all AH algebras with slow dimension growth. As AH algebras without slow dimension growth have been proven to resist K-theoretic classification, such a result would provide a nice tidy resolution to the problem of classifying AH algebras with K-theory.

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